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Power Counting and Wilsonian Renormalization in Nuclear Effective Field Theory

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Effective field theories are the most general tool for the description of low energy phenomena. They are universal and systematic: they can be formulated for any low energy systems we can think of and offer a clear guide on how to calculate predictions with reliable error estimates, a feature that is called power counting. These properties can be easily understood in Wilsonian renormalization, in which effective field theories are the low energy renormalization group evolution of a more fundamental —perhaps unknown or unsolvable— high energy theory. In nuclear physics they provide the possibility of a theoretically sound derivation of nuclear forces without having to solve quantum chromodynamics explicitly. However there is the problem of how to organize calculations within nuclear effective field theory: the traditional knowledge about power counting is perturbative but nuclear physics is not. Yet power counting can be derived in Wilsonian renormalization and there is already a fairly good understanding of how to apply these ideas to non-perturbative phenomena and in particular to nuclear physics. Here we review a few of these ideas, explain power counting in two-nucleon scattering and reactions with external probes and hint at how to extend the present analysis beyond the two-body problem.

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1. Introduction

The structure, properties and reactions of nuclei and nuclear matter depend on the dynamics of the nucleons.¹ This is the reason why the derivation of the nuclear forces is probably the most important problem of nuclear physics. After the discovery of quantum chromodynamics (QCD) — the fundamental theory of strong interactions — a solid theoretical understanding of the nuclear force should be grounded on QCD, either directly or indirectly. Lattice QCD represents the direct, computational derivation: the interaction of quarks and gluons is not analytically solvable at the distances that are characteristic for nuclear physics but it is nu-

merically solvable at the expense of huge computational resources. Recent progress in this front is exciting.^{2–6} The indirect derivation requires to explain the nuclear interaction without explicitly solving QCD. Yet QCD must enter indirectly in the picture. Otherwise we will end up with a phenomenological description instead of a theoretical explanation.

Physics as a science depends to a great extent on the existence of scale separation in nature. One can describe the properties of atoms without explicit knowledge of the composite nature and internal structure of the nuclei within. The nucleus is indeed much smaller than the atom containing it, i.e. there is an excellent separation of scales. Analogously, one can describe the dynamics of nucleons and pions without knowing the details of the strong interaction of the quarks and gluons inside them. However the average distance of nucleons in a nucleus — about $1 - 2$ fm — is not that different from the size of the nucleon or the wavelength of the quarks and gluons inside, maybe 0.5 fm. Without a clear separation of scales the development of satisfactory theoretical explanations to physical phenomena becomes more difficult. As a consequence the description of nuclei is less clear and more involved than that of atoms.

Effective field theories (EFTs) are the standard theoretical tool to exploit the separation of scales of a physical system with the intention of building the most general description of it at low energies.^{7–9} If we call the low energy scale Q and the high energy scale M , an EFT provides a power expansion in terms of Q/M of all the physical quantities of a system. For that one considers first all the possible interaction terms in the Lagrangian that are compatible with the low energy symmetries of the system. Then one orders the infinite Feynman diagrams obtained in the previous step according to their expected size. The method by which we estimate the size of the diagrams is called power counting. While writing the diagrams is trivial, their power counting is not.

The connection of the EFT to the fundamental theory at the scale M is provided by renormalization, the core idea of EFT. In its standard formulation renormalization deals with ultraviolet (UV) divergences in the Feynman diagrams of the EFT. To remove the divergences one includes an UV cut-off and allows the couplings in the Lagrangian to depend on the cut-off. If the calculation of the observable quantities of the EFT is independent of the cut-off then the EFT is renormalizable. Power counting is decided according to how we have to arrange calculations to remove the divergences at each order in the expansion. Wilsonian renormalization¹⁰ provides an alternative but equivalent formulation. Here the starting point is the independence of observables with respect to the cut-off. In this case it is the calculation of the couplings under the assumption of cut-off independence that leads to the size of these couplings at low energies and to their power counting.^{11–14} This is referred to as renormalization group: the focus is on the evolution of the couplings as the cut-off changes, not on the divergences. In Wilsonian renormalization the cut-off runs from the high to the low energy scale, from M to Q . This is counterintuitive from the standard point of view, where the cut-off runs from Q to M .

with the purpose of finding out whether there are UV divergences. Yet they are equivalent. The cut-off can either run to the ultraviolet or the infrared (IR). As far as the observables are independent of the cut-off we end up with identical power countings. The starting point in Wilsonian renormalization can be either an EFT or the fundamental theory. The advantage in the first case is that power counting can be determined without a complete order-by-order calculation of observables. In the second case there is the possibility of evolving a fundamental theory from M to Q , which amounts to uncovering the EFT by means of a concrete calculation. Of course this is only possible in the few cases where the fundamental theory is known or easily solvable (a nice example can be found in Ref. 11). This manuscript is dedicated to Wilsonian renormalization in nuclear EFT:^{12–16} even though it is less well-known than the standard idea of removing divergences, it can provide a clearer interpretation of power counting and the role of the cut-off in EFT.

In nuclear physics the EFT usually contains nucleon and pion fields^a that are constrained by chiral symmetry, a low energy symmetry of QCD that is exact in the limit of massless u , d and sometimes s quarks. The problem is that historically renormalization has been only well understood in the case of systems that are perturbative.^{11, 17} This is the case in hadron physics for processes involving at most one baryon, where chiral perturbation theory (ChPT),¹⁸ the standard EFT for low energy hadronic processes, is used. But in nuclear physics the existence of the deuteron and the virtual state (the 1S_0 singlet), not to mention the few thousand known nuclei, indicates that the nuclear force is non-perturbative. Besides there is the additional problem that EFT entails nuclear forces that are strongly divergent at short distances. Thus it is not a surprise that progress in nuclear EFT has been full of unexpected turns and controversies. Recently we have begun to have a solid grasp of the non-perturbative renormalization of the EFT potentials^{19–23} and how to organize the power counting in this situation,^{14, 24–29} but even these advances have been the subject of debate.^{30–33} Here we will review power counting from the perspective of the renormalization group.

Historically events have unfolded in a zig-zag pattern. Weinberg made the first proposal for a nuclear EFT,^{34, 35} which includes the iteration of the EFT potential (at least at lowest order). This serves to capture the non-perturbative character of nuclear interactions but in exchange requires non-perturbative renormalization. As previously said, this has been the source of a few surprises. Kaplan, Savage and Wise (KSW) discovered a subtle but nonetheless serious inconsistency with the Weinberg proposal.³⁶ These authors also developed a new formulation of nuclear EFT, the KSW counting,^{37, 38} which is free from that inconsistency. However the convergence of the KSW counting in the triplet partial waves happened to be unsatisfying to say the least.³⁹ The community turned back to the Weinberg proposal in search for phenomenological success.^{40, 41} But later Nogga, Timmermans and van Kolck²⁴

^a There are also nuclear EFTs with additional fields – such as the delta isobar – or without the pion field (the pionless EFT).

discovered that the Weinberg proposal contains a new, more conspicuous inconsistency at the lowest order: it is not renormalizable in some P- and D-waves^b. New developments about renormalizability followed^{14, 20–23} that made finally possible a consistent nuclear EFT with good convergence properties.^{25–29} Despite these advancements, there is an ongoing debate about whether these problems are relevant and whether it would be simply more sensible to reinterpret renormalizability for non-perturbative problems in a different way.^{30–33} We will not discuss these new developments, except for a brief comment. Here we are mostly concerned about the derivation of EFT power counting from a specific set of renormalization tools, which happen to be more than enough to make nuclear EFT work at the theoretical level. From this perspective the previous ideas, though interesting, do not appear to be totally necessary.

This manuscript is organized as follows: in Sect. II we introduce Wilsonian renormalization for the particular case of non-relativistic scattering of two particles. Part of it is general and part of it is specific to nuclear physics. We also discuss the relationship of power counting with the anomalous dimension of couplings and the relationship between Wilsonian renormalization and the more standard approach of removing ultraviolet divergences. In Sect. III we extend the results beyond the two-nucleon system, in particular to the deuteron electroweak reactions and to the three-body problem. Finally we summarize our conclusions. We also include an appendix discussing the derivation of a particular equation in this manuscript.

2. Wilsonian Renormalization

Here we illustrate how Wilsonian renormalization works for non-relativistic s-wave scattering.^{12–15} The starting point is a “fundamental theory”. For a non-relativistic two-body system the equivalent of a fundamental theory is the non-relativistic potential V . To obtain the scattering amplitudes we solve the Schrödinger equation at finite momentum k

$$-u_k''(r) + 2\mu V(r) u_k(r) = k^2 u_k(r), \quad (1)$$

where u_k is the reduced wave function, μ the reduced mass, k the center of mass momentum and $V(r)$ the underlying potential, which we assume to be known at all distances. As we are considering s-wave scattering there is no centrifugal term. We solve this equation with the regular boundary condition at the origin

$$u(0) = 0. \quad (2)$$

Finally the phase shift can be extracted from the asymptotic wave function

$$u_k(r) \rightarrow \sin(kr + \delta) \quad \text{for } r \rightarrow \infty. \quad (3)$$

^bActually, the KSW inconsistency already indicated that the Weinberg proposal is not renormalizable at the lowest order. However this problem does not directly affect two-nucleon scattering.

Wilsonian renormalization works as follows. In a first step we include a cut-off r_c as a separation scale

$$V(r) \rightarrow V(r; r_c) = V(r) \theta(r - r_c). \quad (4)$$

We will consider that the physics at distances shorter than the cut-off $r < r_c$ is unknown. Of course if we cut the potential for $r < r_c$ the physical observables will change. We want to prevent this from happening. In a second step we include a new piece in the potential that counteracts the loss of information from having a cut-off and keeps the observables unchanged. This extra piece is the contact-range potential, which can take many parametrizations. For simplicity we choose the following form for the contacts

$$V_C(r; r_c) = \frac{\delta(r - r_c)}{4\pi r_c^2} \sum_{n=0}^{\infty} C_{2n}(r_c) k^{2n}, \quad (5)$$

that is, an energy-dependent delta shell potential. Now we solve the Schrödinger equation with the “renormalized” potential

$$V_R(r; r_c) = V(r; r_c) + V_C(r; r_c). \quad (6)$$

For distances below r_c we have a free Schrödinger equation

$$-u_k''(r) = k^2 u_k(r), \quad (7)$$

with the regular solution

$$u_k(r) = \sin(kr). \quad (8)$$

For distances above r_c we have the original Schrödinger equation, i.e. Eq. (1). Finally at $r = r_c$ the delta-shell potential V_C generates a discontinuity in the first derivative of the wave function that takes the form

$$\frac{u_k'(r_c^+)}{u_k(r_c^+)} - \frac{u_k'(r_c^-)}{u_k(r_c^-)} = \frac{\mu}{2\pi r_c^2} \sum_n C_{2n}(r_c) k^{2n}, \quad (9)$$

where r_c^\pm refers to $r_c \pm \epsilon$, with $\epsilon \rightarrow 0$. A derivation can be found in Appendix A. This is the renormalization group equation (RGE) for the contact-range coupling C_{2n} . The RGE we have written above is exact: the starting point is the full potential V and we want to check what type of contact interaction we have to include to account for the existence of a cut-off radius r_c . For $r_c = 0$ we have the boundary condition $C_{2n}(0) = 0$: we know the potential at all distances and there is no need for the contact-range couplings. As we increase the cut-off radius, we will need non-vanishing $C_{2n}(r_c)$ couplings to account for the missing physics.

The reason we are interested in Wilsonian renormalization is because we want to know how physics looks like at large distances in general. We want to describe phenomena at low energies regardless of which is the fundamental theory at high energies. With Wilsonian renormalization we can build a theory for distances larger than the cut-off ($r \geq r_c$) that is equivalent to the fundamental theory for momenta

$kr_c < 1$. In this context it is useful to define the soft and hard scales Q and M . The soft scale Q is the characteristic momentum of the low energy physics we want to describe, while the hard scale M is the natural momentum scale of the fundamental theory. If we solve the RGE for $Qr_c \rightarrow 1$ we will be able to derive the kind of generic low energy theory we are interested in.

To solve the RGE and obtain the contact range couplings we simply have to make an ansatz for the wave function u_k . The simplest case is provided by a theory in which the underlying potential has a finite range set by the hard scale M

$$V(r) \rightarrow 0 \quad \text{for } Mr \gg 1. \quad (10)$$

With this in mind we see that the wave functions for $Mr \gg 1$ are given by

$$u_k(r) \rightarrow \sin(kr + \delta), \quad (11)$$

where δ is the phase shift of the fundamental potential V . Therefore the solution of the RGE for $Mr_c \gg 1$ is

$$k \cot(kr_c + \delta) - k \cot kr_c = \frac{\mu}{2\pi r_c^2} \sum_n C_{2n}(r_c) k^2. \quad (12)$$

For finding the running of the individual $C_{2n}(r_c)$ couplings we expand the RGE in powers of k^2 . We first take into account that V is a finite-range potential and the effective range expansion applies (for $k < M$)

$$k \cot \delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n}, \quad (13)$$

where a_0 is the scattering length, r_0 the effective range and v_n the shape coefficients. Now we expand and get the set of equations

$$\frac{1}{r_c - a_0} - \frac{1}{r_c} = \frac{\mu}{2\pi r_c^2} C_0(r_c), \quad (14)$$

$$\frac{a_0^2 \frac{r_0}{2}}{(r_c - a_0)^2} + \frac{P_2(r_c, a_0)}{(r_c - a_0)^2} + \frac{r_c}{3} = \frac{\mu}{2\pi r_c^2} C_2(r_c), \quad (15)$$

$$\frac{a_0^2 v_2}{(r_c - a_0)^2} + \frac{P_4(r_c, a_0, r_0)}{(r_c - a_0)^3} + \frac{r_c^3}{45} = \frac{\mu}{2\pi r_c^2} C_4(r_c), \quad (16)$$

plus analogous equations for the higher order couplings. In the equations above $P_2(r_c, a_0)$ and $P_4(r_c, a_0, r_0)$ are polynomials of the cut-off r_c and the effective range coefficients. They are easy to calculate but they are not included here because their exact form is inconsequential for the analysis that follows.

The previous equations are generic solutions for an arbitrary finite range potential V . However the counting of the couplings $C_{2n}(r_c)$ as $Qr_c \rightarrow 1$ depends on which is the size of the effective range coefficients and in particular the scattering length. In general the size of the effective range coefficients is known to scale according to the range of the potential (therefore the name):

$$Mr_0 \sim 1 \quad \text{and} \quad M^{2n+1} v_n \sim 1. \quad (17)$$

The exception is the scattering length a_0 that can take any value, more so if there is non-perturbative physics. Thus we distinguish two possibilities:

$$Ma_0 \sim 1 \quad \text{or} \quad Qa_0 \sim 1. \quad (18)$$

The first one is a scattering length of natural size and the second an unnaturally large scattering length, which is what happens for instance if there is a bound state near the threshold.

If the scattering length is of order $1/M$ we are entitled to expand in powers of a_0/r_c because $Mr_c \gg 1$. We obtain

$$C_0(r_c) = \frac{2\pi}{\mu} a_0 \left[1 + \frac{a_0}{r_c} + \mathcal{O}\left(\frac{a_0^2}{r_c^2}\right) \right], \quad (19)$$

that is, $C_0(r_c)$ scales as $1/M^2$. If we analyze now the subleading couplings $C_{2n}(r_c)$, we find for C_2

$$C_2(r_c) = \frac{2\pi}{\mu} a_0^2 \frac{r_0}{2} \left[1 + 2 \frac{a_0}{r_c} + \mathcal{O}\left(\frac{a_0^2}{r_c^2}\right) \right] + C_2^R(r_c; a_0), \quad (20)$$

that is, C_2 scales as $1/M^4$. In the equation above C_2^R is a “redundant” piece of the coupling C_2 that does not contain information about the effective range r_0 . The function of C_2^R is to absorb the cut-off dependence that the C_0 coupling generates at finite energy. The C_2^R piece of C_2 is inessential for power counting. For C_4 we have

$$C_4(r_c) = \frac{2\pi}{\mu} a_0^2 v_2 \left[1 + 2 \frac{a_0}{r_c} + \mathcal{O}\left(\frac{a_0^2}{r_c^2}\right) \right] + C_4^R(r_c; a_0, r_0), \quad (21)$$

which scales as $1/M^6$ and where C_4^R is analogous to C_2^R , only that it absorbs the residual cut-off dependence of C_0 and C_2 . For the higher order couplings we have $C_{2n} \sim 1/M^{2n+2}$.

The other possibility is that the scattering length is large: $Qa_0 \sim 1$. Now the cut-off and the scattering length can have the same size and we are not allowed to expand in powers of a_0/r_c . If we solve the RGE for C_0 we obtain

$$C_0(r_c) = \frac{2\pi}{\mu} \frac{r_c a_0}{r_c - a_0}, \quad (22)$$

which means $C_0 \sim 1/(MQ)$, an enhancement of one power of M/Q . For the C_2 coupling we get

$$C_2(r_c) = \frac{2\pi}{\mu} a_0^2 \frac{r_0}{2} \frac{r_c^2}{(r_c - a_0)^2} + C_2^R(r_c; a_0), \quad (23)$$

which entails $C_2 \sim 1/(M^2Q^2)$, an enhancement of two powers of M/Q . For the C_4 coupling we find

$$C_4(r_c) = \frac{2\pi}{\mu} a_0^2 v_2 \frac{r_c^2}{(r_c - a_0)^2} + C_4^R(r_c; a_0, r_0), \quad (24)$$

and in general for the C_{2n} we have

$$C_{2n}(r_c) = \frac{2\pi}{\mu} a_0^2 v_n \frac{r_c^2}{(r_c - a_0)^2} + C_{2n}^R(r_c; a_0, r_0, \dots, v_{n-1}), \quad (25)$$

which implies a M^2/Q^2 enhancement over the natural case.

The first implementation of this type of RG analysis of the couplings in nuclear EFT is due to Birse, McGovern and Richardson,¹² who formulated the RGEs in momentum space. Instead of imposing the invariance of the phase shifts with respect to the regulator, their analysis requires the invariance of the full off-shell T-matrix. For contact-range interactions both conditions are equivalent: on-shell renormalization implies off-shell renormalization. Probably this is the case too for finite-range potentials (it has been proved for potentials that have power-law divergences near the origin⁴²). The analysis of the RGEs in momentum space is pretty convoluted though. The analysis of the coordinate space RGEs of Ref. 15 is simpler as it only depends on the Schrödinger equation and its solutions. It connects the RGEs with the cut-off dependence of the observables after an arbitrary number of contact-range operators are included. But at the same time it neglects how the RGEs relate to the power counting of the couplings. The purpose of this section has been to close this gap and to translate the RG analysis of Ref. 12 from momentum to coordinate space, attempting to make its interpretation clearer in the way.

2.1. *Low Energy Effective Field Theory and Power Counting*

The question we wanted to answer is: what kind of low energy theory does one derive from the RGEs? The answer involves two ingredients. The first is a non-relativistic potential for the low energy theory, the effective potential:

$$V_{\text{EFT}}(r; r_c) = V_C(r; r_c) = \frac{\delta(r - r_c)}{4\pi r_c^2} \sum_{n=0}^{\infty} C_{2n}(r_c) k^{2n}, \quad (26)$$

that is, the contact-range potential that compensates the cut-off dependence. The fundamental potential V does not enter into the effective potential for the simple reason that it vanishes at large distances ($Mr \gg 1$). The second ingredient is the size of the couplings, that we have already calculated from the RGEs. As a consequence of the scaling properties of the couplings we can write the effective potential as a power series in Q/M . Let us consider the example of a theory with a natural scattering length, for which we have

$$C_{2n}(r_c) \sim \frac{1}{M^{2n+2}}. \quad (27)$$

If we take into account the typical factors of π and the reduced mass that are common in non-relativistic scattering, the previous scaling allows to rewrite the C_{2n} couplings as

$$C_{2n}(r_c) = \frac{2\pi}{\mu} \frac{c_{2n}(r_c)}{M^{2n+1}}, \quad (28)$$

where $c_{2n}(r_c)$ is a number of $\mathcal{O}(1)$. Now we plug this expression into the effective potential. We arrive to

$$V_{\text{EFT}}(r; r_c) = \frac{\delta(r - r_c)}{4\pi r_c^2} \frac{2\pi}{\mu M} \sum_{n=0}^{\infty} c_{2n}(r_c) \left(\frac{Q}{M}\right)^{2n}, \quad (29)$$

that is, a power series in Q/M . For large scattering length we have instead

$$C_0(r_c) = \frac{2\pi}{\mu Q} c_0(r_c), \quad (30)$$

$$C_{2n}(r_c) = \frac{2\pi}{\mu Q^2} \frac{c_{2n}(r_c)}{M^{2n-1}} \quad \text{for } n \geq 1, \quad (31)$$

with $c_0(r_c) = \mathcal{O}(1)$ and $c_{2n}(r_c) = \mathcal{O}(1)$ for $Qr_c \sim 1$. In this case the expansion of the potential reads

$$V_{\text{EFT}}(r; r_c) = \frac{\delta(r - r_c)}{4\pi r_c^2} \frac{2\pi}{\mu Q} \left[c_0(r_c) + \sum_{n=1}^{\infty} c_{2n}(r_c) \left(\frac{Q}{M}\right)^{2n-1} \right]. \quad (32)$$

Independently of the power counting of the couplings we end up with a series that converges for $Q < M$.

This idea of arranging the effective potential as a power series extends to every physical quantity we can think of. It is a fundamental concept in EFT, the reason why calculations are systematic. We can predict observable quantities up to a given degree of accuracy, that is, up to a given power of the expansion parameter Q/M . The calculations are organized as to only include the operators that contribute within the accuracy goals we have set up in the first place. We can illustrate this concept with the phase shift. In perturbation theory the phase shift is expanded as

$$\begin{aligned} \tan \delta(k) = \frac{2\mu}{k} & \left[\int dr V(r) \sin^2(kr) \right. \\ & + \int dr dr' V(r) \sin(kr) V(r') \sin(kr') G_k(r, r') \\ & \left. + \mathcal{O}(V^3) \right], \end{aligned} \quad (33)$$

i.e. the Born approximation followed by second and higher order perturbation theory. That is, we have written a coordinate space version of the Lippmann-Schwinger equation. In the expression above G_k is a Green function, which we can take to be

$$G_k(r, r') = \frac{2\mu}{k} [\sin kr \cos kr' \theta(r - r') + \cos kr \sin kr' \theta(r' - r)]. \quad (34)$$

If the scattering length is natural, the size of the Born and second order term are

$$\frac{2\mu}{k} \langle V \rangle = \frac{Q}{M} f_1(kr_c) \left[\sum_{n=0}^{\infty} c_{2n}(r_c) \left(\frac{Q}{M}\right)^{2n} \right], \quad (35)$$

$$\frac{2\mu}{k} \langle V G_0 V \rangle = \left(\frac{Q}{M}\right)^2 f_2(kr_c) \left[\sum_{n=0}^{\infty} c_{2n}(r_c) \left(\frac{Q}{M}\right)^{2n} \right]^2, \quad (36)$$

where $\langle V \rangle$, $\langle VG_0V \rangle$ is simply a compact notation for the first and second order of the perturbative series and $f_1(x)$, $f_2(x)$ are functions that encode the cut-off dependence. If we continue we will find that for higher order perturbations we have

$$\frac{2\mu}{k} \underbrace{\langle VG_0 \dots G_0V \rangle}_{r \text{ iterations of } V} = \left(\frac{Q}{M}\right)^r f_r(kr_c) \left[\sum_{n=0}^{\infty} c_{2n}(r_c) \left(\frac{Q}{M}\right)^{2n} \right]^r, \quad (37)$$

where r refers to the number of insertions of the potential V . The EFT expansion for the phase shift starts at Q/M – leading order (LO) from now on – and second order perturbation theory carries an additional factor of Q/M over the Born approximation. Analogously each additional iteration of the potential involves an extra power of Q/M . Putting all the pieces together, the LO calculation only contains C_0 at tree level, the next-to-leading order (NLO) calculation requires to include two iterations of C_0 , the next-to-next-to-leading order (NNLO) calculation contains C_2 at tree level and three iterations of C_0 . Higher orders calculations are set up in a similar fashion.

For the large scattering length case the evaluation of the perturbative series for the tangent of the phase shift leads to

$$\frac{2\mu}{k} \langle V \rangle = f'_1(kr_c) \left[c_0(r_c) + \sum_{n=1}^{\infty} c_{2n}(r_c) \left(\frac{Q}{M}\right)^{2n} \right], \quad (38)$$

$$\frac{2\mu}{k} \langle VG_0V \rangle = f'_2(kr_c) \left[c_0(r_c) + \sum_{n=1}^{\infty} c_{2n}(r_c) \left(\frac{Q}{M}\right)^{2n} \right]^2, \quad (39)$$

$$\frac{2\mu}{k} \underbrace{\langle VG_0 \dots G_0V \rangle}_{r \text{ iterations of } V} = f'_r(kr_c) \left[c_0(r_c) + \sum_{n=1}^{\infty} c_{2n}(r_c) \left(\frac{Q}{M}\right)^{2n} \right]^r. \quad (40)$$

Now the EFT expansion of the phase shift begins at order $(Q/M)^0$, which is the LO for this power counting (i.e. the LO is defined differently for each scaling of the couplings). It is also apparent that the LO calculation contains all the iterations of the C_0 coupling. That is, C_0 is non-perturbative at LO. However all the other couplings are perturbative: C_2 enters at tree level at NLO, C_4 at N³LO and C_{2n} at N²ⁿ⁻¹LO. A systematic exposition of the diagrams involved in the calculation of the amplitudes (for natural and unnatural scattering length) can be found in Ref. 43.

2.2. Renormalization with the One Pion Exchange Potential

The previous analysis can be extended to the possibility that the potential can be separated into a short and long range piece

$$V = V_S + V_L. \quad (41)$$

The range of V_S scales as M , while the range of V_L as Q . To determine the effect of a long range potential on the power counting we follow the previous steps: introduce

a cut-off and include a contact-range potential to keep physics unchanged. The couplings C_{2n} of the contact-range potential can be calculated from the RGE, i.e. Eq. (9). For $Mr_c \gg 1$ the short-range potential vanishes and the wave functions that enter into the RGE are solutions of

$$-u_k''(r) = k^2 u_k(r) \quad \text{for } r < r_c, \quad (42)$$

$$-u_k''(r) + 2\mu V_L(r) u_k(r) = k^2 u_k(r) \quad \text{for } r > r_c, \quad (43)$$

with the boundary conditions

$$u_k(0) = 0, \quad (44)$$

$$u_k(r) \rightarrow \sin(kr + \delta) \quad \text{for } r \rightarrow \infty, \quad (45)$$

with δ the phase shift of the full potential $V_S + V_L$. We are interested in the wave functions in the distance range $M \gg 1/r \geq Q$. The condition $Mr \gg 1$ is necessary if we use a wave function that is a solution of the long range potential V_L . A soft cut-off — let's say $Qr \sim 1$ — is perfectly acceptable. But a excessively soft cut-off of the order of $Qr \ll 1$ is not: for this choice of the cut-off the long range potential V_L vanishes and we end up with the power counting of a pure short range potential.

Everything that is left is to calculate the wave functions for the long range potential. In general the form of the solution of the wave function will take the form

$$u_k(r) = u_a(r; k) + c(k) u_b(r; k), \quad (46)$$

where u_a and u_b are two linearly independent solutions of V_L and $c(k)$ a coefficient that selects the particular linear combination. If we expand the solutions in powers of k^2

$$u_a(r; k) = u_{0,a}(r) + k^2 u_{2,a}(r) + \sum_{n=2}^{\infty} k^{2n} u_{2n,a}(r), \quad (47)$$

$$u_b(r; k) = u_{0,b}(r) + k^2 u_{2,b}(r) + \sum_{n=2}^{\infty} k^{2n} u_{2n,b}(r), \quad (48)$$

and the coefficient $c(k)$ as

$$c(k) = c_0 + c_2 k^2 + \sum_{n=2}^{\infty} c_{2n} k^{2n}, \quad (49)$$

we end up with the following expressions for the running of the C_{2n} couplings

$$\frac{u'_{0a} + c_0 u'_{0b}}{u_{0a} + c_0 u_{0b}} - \frac{1}{r_c} = \frac{\mu}{2\pi r_c^2} C_0(r_c), \quad (50)$$

$$\frac{c_2 u'_{0b}}{u_{0a} + c_0 u_{0b}} + \frac{Q_2(u_{0a}, u'_{0a}, u_{0b}, u'_{0b}, c_0)}{(u_{0a} + c_0 u_{0b})^2} + \frac{r_c}{3} = \frac{\mu}{2\pi r_c^2} C_2(r_c), \quad (51)$$

plus analogous expressions for the higher order couplings, where the wave functions and their derivatives are understood to be evaluated at $r = r_c$. In the expression

above, Q_2 is a polynomial of c_0 and u_{0a} , u_{0b} and its derivatives that encodes the residual cut-off dependence.

A few general comments might be of help at this point. First: the coefficient $c(k)$ can be thought of as the analogous of the ERE in the presence of a long range potential. What this means is that the set of coefficients c_2 , c_4 , etc., will scale according to inverse powers of M . The exception is c_0 , which could take any value if V_S is non-perturbative. Second: if the long range potential is perturbative, the wave functions will coincide with the free wave functions at tree level in perturbation theory. The couplings will also accept a perturbative expansion, but at tree level will coincide with the couplings of the short-range case. Thus the power counting does not change if the long range potential is perturbative.

In nuclear physics the longest range piece of the interaction is the one pion exchange (OPE) potential. This potential can be written as

$$V_{\text{OPE}}(r) = \frac{4\pi}{M_N \Lambda_{\text{NN}}} \frac{m^3}{12\pi} [W_S(r) \vec{\sigma}_1 \cdot \vec{\sigma}_2 + W_T(r) S_{12}(\hat{r})] \vec{\tau}_1 \cdot \vec{\tau}_2, \quad (52)$$

where $\vec{\sigma}_{1(2)}$ and $\vec{\tau}_{1(2)}$ are the spin and isospin operators acting on the nucleon 1(2), m is the pion mass, M_N the nucleon mass and Λ_{NN} is a mass scale that characterizes the strength of the OPE potential (its value is of the order of 300 MeV). The tensor operator is defined as $S_{12} = 3 \vec{\sigma}_1 \cdot \hat{r} \vec{\sigma}_2 \cdot \hat{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2$, while W_S and W_T refer to the spin-spin and tensor components of the potential

$$W_S = \frac{e^{-mr}}{mr}, \quad (53)$$

$$W_T = \frac{e^{-mr}}{mr} \left(1 + \frac{3}{mr} + \frac{3}{(mr)^2}\right). \quad (54)$$

We will ignore the complications coming from the tensor operator and will concentrate on the fundamentals: (i) S_{12} vanishes in the singlet and (ii) W_S and W_T behave as a $1/r$ and a $1/r^3$ potential respectively.

Before analyzing the power counting with OPE it will be helpful to comment on the role of Λ_{NN} . Notice that we have written the OPE potential as

$$V_{\text{OPE}}(r) = \frac{4\pi}{M_N \Lambda_{\text{NN}}} \times P_3\left(m, \frac{1}{r}\right) e^{-mr}, \quad (55)$$

where P_3 is a polynomial that contains a m^2/r , m/r^2 and $1/r^3$ term, where all the terms have three powers of $Q = \{m, 1/r\}$. The analogy with the contact-range potential is clear, more so if we write V_C as

$$V_C(r; r_c) = C_0(r_c) \times \frac{1}{4\pi r^3} \delta\left(1 - \frac{r}{r_c}\right), \quad (56)$$

where we can appreciate that the $4\pi/(M_N \Lambda_{\text{NN}})$ factor in V_{OPE} plays the same role as the C_0 coupling in V_C . That is, if we count Λ_{NN} as M the OPE potential will be perturbative.^{37,38} On the contrary if we count Λ_{NN} as Q the OPE potential will be non-perturbative.¹⁴ We are interested in the later case: as we have already pointed out, if OPE is perturbative the counting is the same as that of a pure

short-range potential. The bottom line is that non-perturbative OPE goes along with the assumption that Λ_{NN} is a light scale. The idea that non-perturbative OPE requires the existence of an additional light scale (besides the obvious choices such as the external momenta, the pion mass or the inverse of the scattering length) was probably *explicitly* realized in Ref. 8 for the first time. In Ref. 14 one can see how to include this scale in the RG equations and what kind of consequence it has for the power counting.

The $1/r$ potential, which corresponds to the OPE potential in the 1S_0 singlet, is the easiest to analyze. Here we are not going to enter into the specific details of how to do the detailed analysis. We merely comment that the power counting is unchanged with respect to the case where there is no long range potential.¹³ That is, there are two possible arrangements of the power counting: a natural one, in which the couplings scale as $C_{2n} \sim 1/M^{2n+2}$ and an unnatural one, in which the couplings scale as $C_0 \sim 1/(MQ)$ and $C_{2n} \sim 1/(Q^2 M^{2n})$. The reason for that is that the $1/r$ potential is not strong enough as to modify the behaviour of the wave functions in the distance window $M \ll 1/r \ll \Lambda_{\text{NN}}$. Even if the strength of the potential is such as to generate a low lying bound state, the wave functions are only substantially modified for $\Lambda_{\text{NN}} \ll 1/r \ll m$. However this cut-off range is not of interest for power counting because we are already making the assumption that $\Lambda_{\text{NN}} \sim m \sim Q$.

For the 3S_1 triplet the potential behaves as an attractive $1/r^3$ for $mr < 1$, which induces important changes in the scaling of the couplings^c. The first thing to notice is that there is not anymore a natural and unnatural power counting. The solutions of the wave function are all equally fine-tuned: there is not a more natural or preferred solution (see Ref. 14 for a different conclusion though). The reason is that the attractive $1/r^3$ potential has no unique solution in quantum mechanics: the choice of the solution inherently depends on the existence of short range physics, which is the only responsible for fixing the wave function. Every linear combination of independent wave functions is equally acceptable. For $mr < 1$ the wave function can be written as^{20–22}

$$u_k(r) \propto r^{3/4} \sin \left(\frac{\beta}{\sqrt{\Lambda_{\text{NN}} r}} + \phi_3(k) \right) \times \left[1 + \mathcal{O}(\sqrt{\Lambda_{\text{NN}} r}, k^2 r^2, mr) \right], \quad (57)$$

where β is a dimensionless number and $\phi_3(k)$ is a phase – the semiclassical phase – that characterizes the particular solution we are dealing with. The value of $\phi_3(k)$ depends on the short-range physics. The changes in the counting are the following:

$$C_0(r_c) \sim \frac{1}{MQ}, \quad (58)$$

$$C_{2n}(r_c) \sim \frac{1}{M^{2n+3/2} Q^{1/2}}. \quad (59)$$

^cActually this is a simplification: the $^3S_1 - ^3D_1$ triplet is a coupled channel and the OPE is a matrix: the tensor operator contains an attractive and repulsive eigenvalue, and the attractive one happens to have a bigger impact on power counting.

This result requires a careful examination of the scaling properties of $\phi_3(k)$, which are not trivial (the derivation is not contained here but will be included in a future publication). As a matter of fact the counting with an attractive tensor force is more similar to NDA than to that of a short range potential with large scattering length.

For a repulsive singular potential the analysis is analogous with the exception of a few details. The wave function is

$$u_k(r) \propto r^{3/4} \left[\exp\left(-\frac{\beta}{\sqrt{\Lambda_{\text{NN}} r}}\right) + c_3(k) \exp\left(+\frac{\beta}{\sqrt{\Lambda_{\text{NN}} r}}\right) \right] \times \left[1 + \mathcal{O}(\sqrt{\Lambda_{\text{NN}} r}, k^2 r^2, mr) \right], \quad (60)$$

with $c_3(k)$ a coefficient that depends on the short-range physics. It is expected to be small and as happened with ϕ_3 its scaling properties are important in the detailed analysis, yet they are not trivial. The scaling of the coupling now is

$$C_0(r_c) \sim \frac{1}{M^{3/2} Q^{1/2}}, \quad (61)$$

$$C_{2n}(r_c) \sim \frac{1}{M^{2n+3/2} Q^{1/2}}. \quad (62)$$

That is, the only difference with the attractive case is the scaling of the C_0 coupling. However the previous scaling properties are difficult to interpret. It is sensible to expect that the importance of short-range physics within EFT depends on the long-range dynamics. The attractive $1/r^3$ potential complies with this expectation: as a consequence of the strong attraction the wave function is enhanced at short distances, which in turn enhances the short-range couplings. For the repulsive $1/r^3$ potential we expect the contrary to happen, that the size of the C_{2n} couplings diminishes. What happens is precisely the contrary, which is puzzling to say the least.

With this we have finished the discussion of power counting for the moment. The types of power counting and the physical situations to which they correspond are summarized in Table 2.2. Of course they are not the only types of power counting that can be built, but they are for sure the most relevant ones for nuclear EFT. Now I will try to show how to rederive these counting rules with other methods. In particular I will consider the calculation of anomalous dimensions, ultraviolet renormalizability and residual cut-off dependence.

2.3. Power Counting and Anomalous Dimensions

There is a very interesting simplification in the above calculations: it is enough to take into account the cutoff dependence of C_{2n} to guess its scaling.¹⁶ More specifically we refer to the cut-off dependence for $\frac{r_c}{a_0} \leq 1$ ($Qr_c \leq 1$), a condition that will remarkably simplify the discussion below. If we consider a two-body system

Table 1. Power counting in the s-wave two-body system

$V_L = 0$ or $V_L \sim Q^0$ ($1/r$ or $1/r^3$ type) or $V_L \sim Q^{-1}$ ($1/r$ type)				
	$a_0 \sim 1/M$	$a_0 \sim 1/Q$	$V_L \sim -1/r^3$	$V_L \sim +1/r^3$
C_0	Q^0 (LO)	Q^{-1} (LO)	Q^{-1} (LO)	$Q^{-1/2}$ (LO)
$C_2 k^2$	Q^2 (N^2 LO)	Q^0 (NLO)	$Q^{3/2}$ ($N^{5/2}$ LO)	$Q^{3/2}$ (N^2 LO)
$C_4 k^4$	Q^4 (N^4 LO)	Q^2 (N^3 LO)	$Q^{7/2}$ ($N^{9/2}$ LO)	$Q^{7/2}$ (N^4 LO)
\dots	\dots	\dots	\dots	\dots
$C_{2n} k^{2n}$	Q^{2n} (N^{2n} LO)	Q^{2n-2} (N^{2n-1} LO)	$Q^{2n-1/2}$ ($N^{2n+1/2}$ LO)	$Q^{2n-1/2}$ (N^{2n} LO)
δC_0	Q^0 (stable)	Q^{-2} (unstable)	$Q^{-1/2}$ (stable)	$Q^{-1/2}$ (stable)

Note: Summary of the power counting for s-wave two-body scattering. The table indicates when the coupling enters as a power of Q (and the relative order in parenthesis). In the text we have considered the case of a pionless and a pionful EFT. For pionless the scaling of the couplings depends on the size of the scattering length. For pionful the scaling is identical to the pionless case if either one of these conditions is met: (i) pion exchanges are perturbative (ii) pion exchanges are non-perturbative but there is only the central piece. If the tensor piece is non-perturbative the scaling of the couplings will be modified with respect to the previous cases. We show this in the table by indicating whether the long-range potential is zero ($V_L = 0$), perturbative ($V_L \sim Q^0$) or non-perturbative ($V_L \sim Q^{-1}$) and then the type of long-range potential ($1/r$ or $1/r^3$). Finally in the last row we indicate the size of a perturbation of C_0 , which determines whether the power counting is infrared stable or unstable (see discussion around Eq. 104)

with natural scattering length the cut-off dependence of the C_{2n} couplings is trivial

$$C_0(r_c) = \frac{2\pi}{\mu} a_0 \times \left[1 + \mathcal{O}\left(\frac{r_c}{a_0}\right) \right], \quad (63)$$

$$C_{2n}(r_c) = \frac{2\pi}{\mu} a_0^2 v_n \times \left[1 + \mathcal{O}\left(\frac{r_c}{a_0}\right) \right], \quad \text{for } n \geq 1, \quad (64)$$

while for a system with a large scattering length we have

$$C_0(r_c) = -\frac{2\pi}{\mu} r_c \times \left[1 + \mathcal{O}\left(\frac{r_c}{a_0}\right) \right], \quad (65)$$

$$C_{2n}(r_c) = \frac{2\pi}{\mu} r_c^2 v_n \times \left[1 + \mathcal{O}\left(\frac{r_c}{a_0}\right) \right] \quad \text{for } n \geq 1, \quad (66)$$

where to simplify the notation we have taken $v_1 = r_0/2$. That is, the power-law dependence on the cut-off matches the enhancement of the coupling. The rule is simple: if $C_{2n}(r_c) \propto r_c^\alpha$ for $M \ll 1/r_c \ll Q$ the size of C_{2n} for $Qr_c \rightarrow 1$ is $1/(M^{2n-\alpha}Q^\alpha)$, a M^α/Q^α enhancement. Equivalently, in momentum space, if $C_{2n}(\Lambda) \propto \Lambda^{-\alpha}$ for $M \ll \Lambda \ll Q$ the size of C_{2n} for $\Lambda \rightarrow Q$ is $1/(M^{2n-\alpha}Q^\alpha)$.

Why is that so? Actually the idea can be better explained with a momentum space cut-off. If the couplings scale as

$$C_{2n}(r_c) \propto r_c^\alpha, \quad (67)$$

with respect to the radial cut-off r_c , in momentum space they will scale as ^d

$$C_{2n}(\Lambda) \propto \Lambda^{-\alpha}, \quad (68)$$

which simply amounts to take into account that $\Lambda \propto 1/r_c$. This scaling property implies that the couplings follow a RGE of the type

$$\frac{d}{d\Lambda} [\Lambda^\alpha C_{2n}(\Lambda) + \dots] = 0, \quad (69)$$

where the dots refer to corrections involving smaller powers of Λ . For the moment we will assume that this RGE is valid in the cut-off window $M \geq \Lambda \geq Q$. If we ignore the dots the solution is straightforward

$$\Lambda_1^\alpha C_{2n}(\Lambda_1) = \Lambda_2^\alpha C_{2n}(\Lambda_2), \quad (70)$$

with Λ_1 and Λ_2 two arbitrary cut-offs. Therefore with a boundary condition we can get the running of the couplings for arbitrary Λ . This boundary condition is the value of the couplings at $\Lambda = M$. At this scale we expect the couplings to scale with M (we do not expect Q to play a role at high energies, which means that M is the only relevant scale), which implies

$$C_{2n}(M) \propto \frac{1}{M^{2n+2}}. \quad (71)$$

As a consequence

$$C_{2n}(Q) \propto \frac{1}{M^{2n+2}} \times \left(\frac{M}{Q}\right)^\alpha, \quad (72)$$

which is the expected enhancement for C_{2n} . In short, the scaling of the coupling decides the power counting. This idea is not new and has appeared in different contexts. In the KSW counting^{37,38} the hard scale can be deduced from the running of the $C_0(\Lambda)$ coupling: the scaling of $C_0(\Lambda)$ changes when Λ approaches Λ_{NN} , which happens to be the hard scale in KSW ^e. Recently it has been applied in nuclear EFT for the analysis of reactions on the deuteron.¹⁶

There is the issue of where the RGE of the C_{2n} couplings comes from, which is related to the calculation of the power α that appears in it. The starting point in Wilsonian renormalization is to include a cut-off and then require observable quantities to be independent of the cut-off

$$\frac{d}{d\Lambda} \langle \Psi | \mathcal{O} | \Psi \rangle = 0, \quad (73)$$

where $|\Psi\rangle$ is the wave function and \mathcal{O} an operator corresponding to an observable. Notice that here we are demanding the matrix element to be independent of the cut-off. Actually this condition is too strong — observables are the square modulus

^dWe use the same notation for the couplings in coordinate and momentum space: we indicate which one we are dealing with by the argument: r_c or Λ .

^eIn KSW the regularization scale is normally referred as μ instead of Λ . It is also worth noticing that KSW does not use a cut-off regularization, but a variant of dimensional regularization.

of matrix elements — but in most situations it will work ^f. If we are in the cut-off window $M \geq \Lambda \geq Q$ we can substitute the wave function and the operator by the corresponding ones in the EFT

$$|\Psi\rangle = |\Psi_{\text{EFT}}\rangle, \quad (74)$$

$$\mathcal{O} = \mathcal{O}_{\text{EFT}}. \quad (75)$$

Moreover the operator \mathcal{O}_{EFT} can be divided into a contact- and finite-range piece

$$\mathcal{O}_{\text{EFT}} = \mathcal{O}_C + \mathcal{O}_F. \quad (76)$$

Now we can rewrite

$$\frac{d}{d\Lambda} \langle \Psi_{\text{EFT}} | \mathcal{O}_C | \Psi_{\text{EFT}} \rangle = -\frac{d}{d\Lambda} \langle \Psi_{\text{EFT}} | \mathcal{O}_F | \Psi_{\text{EFT}} \rangle, \quad (77)$$

which tell us that the contact-range piece has two functions: to absorb the cut-off dependence of the finite-range piece and to directly contribute to the matrix element. Had we used the full wave function $|\Psi\rangle$ and the full operator \mathcal{O} instead of the EFT ones, the contact would have only been there to absorb the cut-off dependence (the contacts vanish for $\Lambda \rightarrow \infty$). But within the EFT description the contacts must have a non-trivial contribution to the observables regardless of the cut-off. The reason is that the finite-range piece of the EFT potential/wave function/operator does not correspond to the fundamental potential/wave function/operator. The bottom line is that for the contact-range operators we can distinguish between a piece that directly contributes to observables and a piece that absorbs cut-off dependence

$$\mathcal{O}_C = \mathcal{O}_C^D + \mathcal{O}_C^R, \quad (78)$$

where the superscript D and R stand for “direct” and “residual”. This distinction is analogous to the one that we made previously for the running of the C_{2n} in short-range theories. Each of these pieces follows a different RGE

$$\frac{d}{d\Lambda} \langle \Psi_{\text{EFT}} | \mathcal{O}_C^D | \Psi_{\text{EFT}} \rangle = 0, \quad (79)$$

$$\frac{d}{d\Lambda} \langle \Psi_{\text{EFT}} | \mathcal{O}_C^R | \Psi_{\text{EFT}} \rangle = -\frac{d}{d\Lambda} \langle \Psi_{\text{EFT}} | \mathcal{O}_F | \Psi_{\text{EFT}} \rangle, \quad (80)$$

corresponding to their different roles within EFT. Notice that we are assuming that the distinction between \mathcal{O}^D and \mathcal{O}^R exists. This is not clear (if we want the definitions to be unambiguous), but we are only using this distinction to simplify the arguments here. We can write a contact-range operator \mathcal{O}_C^D as a coupling times a polynomial involving the light scales in momentum space

$$\mathcal{O}_C^D = C(\Lambda) \times P_\Lambda(Q), \quad (81)$$

^f An example where the phase is important can be found in the infrared renormalization of Coulomb in proton-proton scattering of Ref. 44.

where $C(\Lambda)$ is the coupling and P_Λ is the polynomial, which can be regularized (hence the subscript Λ). If we include this general form in the RGE for the “direct” piece we arrive to

$$\frac{d}{d\Lambda} [C(\Lambda) \langle \Psi_{\text{EFT}} | P_\Lambda(Q) | \Psi_{\text{EFT}} \rangle] = 0. \quad (82)$$

What is left is to determine the cut-off dependence of the matrix element of the polynomial, which in general will take the form

$$\langle \Psi_{\text{EFT}} | P_\Lambda(Q) | \Psi_{\text{EFT}} \rangle \propto \Lambda^\alpha \times \left[1 + \mathcal{O}\left(\frac{Q}{\Lambda}, \frac{\Lambda}{M}\right) \right], \quad (83)$$

where the form of the corrections follow from the assumption that the RGE are valid in the region $M \geq \Lambda \geq Q$, that is, from the analyticity of the RGE which in turn implies that we can write a power series on Q/Λ and Λ/M .

The previous discussion is rather general and concerns any observable that receives a direct, linear contribution from a contact-range operator. In the case of two-body scattering the matrix element we are interested in is the T-matrix

$$\frac{d}{d\Lambda} \langle k | T_{\text{EFT}} | k' \rangle = 0, \quad (84)$$

which is not receiving a linear contribution from the contact-range physics, at least at first sight. The T-matrix is the solution of the Lippmann-Schwinger equation

$$T_{\text{EFT}} = V_{\text{EFT}} + V_{\text{EFT}} G_0 T_{\text{EFT}}, \quad (85)$$

which is a convenient way of rewriting the Schrödinger equation for a scattering problem. In EFT we can expand the T-matrix and the potential as power series

$$T_{\text{EFT}} = \sum_{\nu} \left(\frac{Q}{M} \right)^{\nu} \hat{T}^{(\nu)}, \quad (86)$$

$$V_{\text{EFT}} = \sum_{\nu} \left(\frac{Q}{M} \right)^{\nu} \hat{V}^{(\nu)}, \quad (87)$$

or more concisely as

$$T_{\text{EFT}} = T_{\text{LO}} + \delta T_{\text{EFT}}, \quad (88)$$

$$V_{\text{EFT}} = V_{\text{LO}} + \delta V_{\text{EFT}}, \quad (89)$$

that is, a LO contribution plus a subleading correction. The interesting thing here is that the subleading correction to the T-matrix is perturbative and is given by

$$\delta T_{\text{EFT}} = \langle k | (1 + T_{\text{LO}} G_0) \delta V_{\text{EFT}} (G_0 T_{\text{LO}} + 1) | k' \rangle + \mathcal{O}[(\delta V_{\text{EFT}})^2] \quad (90)$$

$$= \langle \Psi_{\text{LO}} | \delta V_{\text{EFT}} | \Psi_{\text{LO}} \rangle + \mathcal{O}[(\delta V_{\text{EFT}})^2], \quad (91)$$

where in the second line $|\Psi_{\text{LO}}\rangle$ is the LO wave function ($|\Psi_{\text{EFT}}\rangle = |\Psi_{\text{LO}}\rangle + \delta |\Psi_{\text{LO}}\rangle$). If we ignore the iteration of δV_{EFT} , make the separations

$$\delta V_{\text{EFT}} = \delta V_C + \delta V_F, \quad (92)$$

$$\delta V_C = \delta V_C^D + \delta V_C^R, \quad (93)$$

and follow the steps previously described, we end up with the RGE

$$\frac{d}{d\Lambda} \langle \Psi_{\text{LO}} | \delta V_C^D | \Psi_{\text{LO}} \rangle = 0. \quad (94)$$

The solution of this RGE depends on the evaluation of the matrix element of the contact-range potential. The details depend on the particular representation for the contacts. For a delta-shell representation in coordinate space

$$V_C(r; r_c) = \frac{\delta(r - r_c)}{4\pi r_c^2} \sum_n C_{2n}(r_c) k^{2n}, \quad (95)$$

the evaluation is trivial

$$\langle \Psi_{\text{LO}} | V_C | \Psi_{\text{LO}} \rangle = \frac{u_k^2(r_c)}{4\pi r_c^2} \sum_{n=0}^{\infty} C_{2n}(r_c) k^{2n}, \quad (96)$$

with u_k the LO reduced wave function. We can concentrate on the evaluation of a particular coupling C_{2n} , in which case we obtain

$$\langle \Psi_{\text{LO}} | C_{2n}(r_c) k^2 | \Psi_{\text{LO}} \rangle = \frac{u_0^2(r_c)}{4\pi r_c^2} C_{2n}(r_c) k^{2n} + \mathcal{O}(k^2), \quad (97)$$

where u_0 is the zero-energy LO reduced wave function. The RGE for the coupling C_{2n} reads

$$\frac{d}{dr_c} \left[\frac{u_0^2(r_c)}{r_c^2} C_{2n}(r_c) + \dots \right] = 0, \quad (98)$$

which means that the running is determined by the power-law dependence of the wave function for $Qr \leq 1$.

In a purely short-range theory the running of the C_{2n} couplings is easy to compute. The zero-energy wave function is

$$u_0(r) = \mathcal{A} (r - a_0), \quad (99)$$

where \mathcal{A} is a normalization factor that is arbitrary (it does not affect the running of C_{2n}). We remind the reader that we are interested in the region $1/M \geq r \geq 1/Q$. If the scattering length is natural ($Ma_0 \sim 1$), we can take $\mathcal{A} = 1$ and rewrite the wave function as

$$u_0(r) = r \times \left[1 + \mathcal{O}\left(\frac{1}{Mr}\right) \right]. \quad (100)$$

Therefore the RGE for the couplings is

$$\frac{d}{dr_c} [C_{2n}(r_c) + \dots] = 0, \quad (101)$$

As a consequence $C_{2n}(Q) \sim 1/M^{2n+2}$, in agreement with the previous determination. If the scattering length is large ($Qa_0 \sim 1$), we set the normalization to $\mathcal{A} = 1/a_0$ to express the wave function as

$$u_0(r) = 1 + \mathcal{O}(Qr), \quad (102)$$

which leads to the RGE

$$\frac{d}{dr_c} \left[\frac{C_{2n}(r_c)}{r_c^2} + \dots \right] = 0. \quad (103)$$

That is, the couplings scale as $C_{2n}(Q) \sim 1/(M^{2n}Q^2)$.

There is a point to explain here: the counting of C_0 can not always be determined with this method. The reason is that we are calculating the scaling of the perturbative piece of the potential. If C_0 is perturbative in the first place we will obtain the correct scaling. This is the case in a short-range theory with natural scattering length, where we get $C_0 \sim 1/M^2$. On the contrary if C_0 is non-perturbative the ideas presented here do not apply. We know that C_0 is enhanced by M/Q if the scattering length is large. Yet the application of this method to C_0 is not useless: it gives us information about the scaling of a small, perturbative change of C_0

$$C_0 \rightarrow C_0 + \delta C_0, \quad (104)$$

where δC_0 is enhanced as M^2/Q^2 . That is, the perturbation δC_0 is of lower order than the original unperturbed coupling C_0 . How is that so? The meaning of this enhancement for δC_0 is that systems with large scattering lengths are fine-tuned.¹² A minor change in C_0 generates a large change in the scattering length. In particular

$$\delta C_0 = \frac{2\pi}{\mu} \frac{r_c^2}{(r_c - a_0)^2} \delta a_0, \quad (105)$$

which entails $\delta a_0 \propto a_0^2 \delta C_0$. From the RG flow perspective the natural solution represents a stable fixed point of the RG equations and the large scattering length solution an unstable fixed point.¹² That is, the running of the C_0 coupling eventually behaves as a constant as $r_c \rightarrow \infty$. But if the scattering length is large C_0 will scale as r_c^2 only as far as $a_0/r_c \geq 1$. The scaling of δC_0 for the different cases that we are considering can be consulted in Table 2.2.

The extension to the pionful EFT is trivial but requires a case-by-case discussion. If pion exchanges are perturbative (i.e. subleading), the power counting is exactly the same as in the short-range case. The reason is that the LO wave functions are identical to the short-range case. If pion exchanges are non-perturbative (i.e. leading), the power counting depends on whether the potential is $1/r$ (central) or $1/r^3$ (tensor). For central OPE the power counting is again as for a short-range potential because the wave functions behave either as 1 or as r for $Qr \leq 1$. However for attractive tensor OPE the power counting changes. Using the wave function written in Eq. 57 we find

$$\frac{d}{dr_c} \left[\frac{C_{2n}(r_c)}{r_c^{1/2}} \sin^2 \left[\sqrt{\frac{\beta}{\Lambda_{\text{NN}} r}} + \phi_3 \right] + \dots \right] = 0, \quad (106)$$

where the dots account for power-law corrections, which are at least of order $\sqrt{m r}$ ($\sqrt{Q r}$). It is worth noting that the corrections to the wave function were computed in Ref. 20 for the ${}^3S_1 - {}^3D_1$ triplet. It can be checked that they do not affect the

counting. The conclusion is that the C_{2n} are bigger than expected by a factor of $\sqrt{M/Q}$.

The title of this section makes mention of the *anomalous dimension*. What is that? The concept is easy to understand. Let us assume that we have a physical quantity (operator, coupling, observable)

$$A = A(Q, \Lambda, M), \quad (107)$$

where A depends on the light scale Q , on the cut-off Λ and the hard scale M . We can define several types of dimensions for A . The most obvious one is the canonical dimension d , which can be related to the rescaling

$$A(\lambda Q, \lambda \Lambda, \lambda M) = \lambda^d A(Q, \Lambda, M), \quad (108)$$

that is, the canonical dimension refers to how A changes with a change of physical units. Another type of dimension we can define is the power counting dimension, which refers to a rescaling of Q (and Λ) only

$$A(\lambda Q, \lambda \Lambda, M) = \lambda^\nu A(Q, \Lambda, M). \quad (109)$$

It is important to notice that while the canonical dimension of a physical quantity is unique, the power counting dimension is not. Rather a physical quantity is a superposition of contributions with different power counting dimensions

$$A = \sum A^{(\nu)} \quad \text{where} \quad A^{(\nu)}(\lambda Q, \lambda \Lambda, M) = \lambda^\nu A^{(\nu)}(Q, \Lambda, M). \quad (110)$$

The inclusion of Λ among the things we rescale for the power counting dimension seems counter-intuitive at first, but it is natural once we consider the argument about the RG evolution of cut-off dependent quantities from $\Lambda = M$ to $\Lambda = Q$. Finally the anomalous dimension can be defined as

$$A(Q, \lambda \Lambda, M) = \lambda^a A(Q, \Lambda, M), \quad (111)$$

which is exactly the kind of power-law dependence on the cut-off that we have been studying along this section. Thus we can restate that the anomalous dimension of a coupling is what determines its power counting.

2.4. Ultraviolet Renormalizability

Wilsonian renormalization is not the most popular or widely understood method of analyzing power counting in EFTs. This honor corresponds to ultraviolet (UV) renormalizability, in which contact-range couplings are included to absorb divergences in Feynman diagrams. Quantum electrodynamics (QED) provides a good illustration of this idea for a quantum field theory (QFT) that only contains marginal or relevant operators[§], i.e. what is traditionally known as a *renormalizable* QFT.

[§]The importance of a relevant (irrelevant) operator grows (decreases) at low energies, while the size of a marginal operator remains approximately the same regardless of energy.⁴⁵ In the RGA of Refs. 12, 13, 14 and also of this manuscript, the previous classification translate into the following:

At this point it is important to mention that nowadays — after the discovery of EFTs — renormalizability is understood in a broader sense. Yet for EFTs the application of this principle is simple: we begin by considering the matrix element of an EFT operator between EFT wave functions

$$\langle \Psi_{\text{EFT}} | \mathcal{O}_{\text{EFT}} | \Psi_{\text{EFT}} \rangle. \quad (112)$$

The operator contains a finite- and a contact-range piece. For the moment we will ignore the contact-range piece because we want to use these operators to remove divergences in the finite-range piece. Thus we consider

$$\langle \Psi_{\text{EFT}} | \mathcal{O}_{\text{F}} | \Psi_{\text{EFT}} \rangle. \quad (113)$$

Now we expand this matrix element in powers of Q/M as before. To simplify the analysis we only take into account the LO wave functions

$$\sum_{\nu} \langle \Psi_{\text{LO}} | \mathcal{O}_{\text{F}}^{(\nu)} | \Psi_{\text{LO}} \rangle + \mathcal{O}(\delta \Psi^{(\nu)}), \quad (114)$$

where we expect the contributions coming from the subleading corrections to the wave function to be inessential for the analysis. In the final step we isolate the ν -th order contribution, include a cut-off Λ and check whether the matrix element

$$\langle \Psi_{\text{LO}} | \mathcal{O}_{\text{F}}^{(\nu)} | \Psi_{\text{LO}} \rangle_{\Lambda} \quad (115)$$

is finite for $\Lambda \rightarrow \infty$. If not, we include contact-range contributions until the matrix element

$$\lim_{\Lambda \rightarrow \infty} \langle \Psi_{\text{LO}} | \mathcal{O}_{\text{F}}^{(\nu)} + \mathcal{O}_{\text{C}}^{(\nu)} | \Psi_{\text{LO}} \rangle_{\Lambda} \quad (116)$$

is finite. If a divergences requires the inclusion of a new contact at order ν , the contact is counted as being of this order.

We can illustrate the idea in non-relativistic scattering, where the relevant matrix element is

$$\langle \Psi_{\text{LO}} | V_{\text{F}}^{(\nu)} | \Psi_{\text{LO}} \rangle = \int_{r_c} dr V_{\text{F}}^{(\nu)}(r) u_k(r)^2, \quad (117)$$

where $u_k(r)$ represents the LO reduced wave function and r_c ($\propto 1/\Lambda$) is the radial cut-off. In the formula above a contribution to the finite-range potential is said to be of order ν when it contains ν powers of the light scales in the momentum space representation

$$\langle p' | V_{\text{F}}^{(\nu)} | p \rangle \propto \frac{Q^{\nu}}{M^{\nu+2}} f\left(\frac{Q}{Q'}\right), \quad (118)$$

where Q includes p , p' , the pion mass m in pionful nuclear EFT and/or other scales depending on the particular EFT we are dealing with. The expression Q/Q' refers

for a relevant (irrelevant) operator the running of its coupling behaves as a negative (positive) power of Λ for $M > \Lambda > Q$, while for a marginal operator the coupling runs either as a constant, as $\log \Lambda$ or more generally as something that is not power-law.

to an arbitrary ratio of light scales (for instance, p/m and p'/m in nuclear EFT) and f is a non-polynomial function that we must compute from the EFT Lagrangian, but which exact form is not important at this point. If we Fourier-transform this expression into coordinate space (and assume for simplicity that the potential is local), we find

$$V_F^{(\nu)}(r) \propto \frac{1}{M^{\nu+2} r^{\nu+3}} f'\left(\frac{Q}{Q'}\right), \quad (119)$$

where Q/Q' refers to mr in nuclear physics. The point is that we know the UV behaviour of the EFT potential. Provided we have the LO wave functions we can analyze the matrix element

$$\langle \Psi_{\text{LO}} | V_F^{(\nu)} | \Psi_{\text{LO}} \rangle \propto \int_{r_c} dr \frac{u_k(r)^2}{r^{\nu+3}}, \quad (120)$$

for divergences and decide which contacts to include.

The complete analysis can be found in Refs. 25, 26. Here we merely comment on the results. If the LO wave function comes from the $1/r$ potential, perturbative and Wilsonian renormalization lead to identical power countings. This is also true if the LO wave function comes from a purely contact-range potential. On the contrary if the LO potential is of the $1/r^3$ type and attractive there is a small, yet significant difference between perturbative and Wilsonian renormalization. Removing the divergences only requires the C_{2n} to enter at order $\nu = (5n - 1)/2$, in contrast with $\nu = 2n - 1/2$ from RGE. The apparent scaling of the couplings is thus $C_{2n} \sim Q^{(n-1)/2}/M^{(5n+3)/2}$, i.e. an extra suppression of $(Q/M)^{n/2}$ with respect to the Wilsonian renormalization value $C_{2n} \sim 1/(M^{2n+3/2}Q^{1/2})$. If the LO $1/r^3$ potential is repulsive the matrix elements for scattering are always finite and no contact interaction is required. However we will not discuss this problem here. Back to the attractive $1/r^3$ potential the reason for the mismatch probably has to do with the k^2 expansion of the LO wave function, which induces a contamination of $(\Lambda_{\text{NN}})^{n/2}$ into the C_{2n} coupling. In fact the k^2 expansion of the LO wave function reads

$$u_k(r) = r^{3/4} \sin\left(\frac{\beta}{\sqrt{\Lambda_{\text{NN}}r}} + \phi_3\right) \times \left[c_0 + c_2 (kr)^2 \sqrt{\Lambda_{\text{NN}}r} + c_4 (kr)^4 (\sqrt{\Lambda_{\text{NN}}r})^2 + \dots\right] \quad (121)$$

with β , c_0 , c_2 , c_4 , etc. numerical coefficients and where ϕ_3 now is independent of energy^h. We can see that each two powers of k imply half a power of Λ_{NN} . As a consequence the couplings that make the matrix element of the potential finite are

^hWhen we wrote the $1/r^3$ wave functions for the RG equations we included an energy-dependent semiclassical phase (see Eq. 57), instead of an energy-independent one like here. The reason is that here we are writing the LO wave functions, in which only the energy-independent C_0 operator contributes, while there we were writing the generic solution of the $1/r^3$ potential with arbitrary short-range physics.

not the standard C_{2n} 's but implicitly contain n half integer powers of Λ_{NN} . That is, they have a different operator structure: $\Lambda_{\text{NN}}^{n/2} k^{2n}$ instead of k^{2n} . In the same way that it is useful to make the distinction

$$C_0 \quad \text{versus} \quad D_2 m_\pi^2, \quad (122)$$

we can also write

$$C_{2n} k^{2n} \quad \text{versus} \quad E_{2n} k^{2n} \Lambda_{\text{NN}}^{n/2}, \quad (123)$$

to make the different structure of these couplings explicit. In this notation the E_{2n} 's happen to be enhanced by $Q^{1/2}$, just as the C_{2n} 's. However the drawback of this explanation is that unlike the D_2 coupling, the proposed E_{2n} couplings do not have a clear interpretation at the lagrangian level.

2.5. *Residual Cut-off Dependence*

The analysis of the residual cut-off dependence of the matrix elements is another method for determining the power counting.^{27–29} First we will review the theoretical basis for this idea: for that we consider a matrix element for which all UV divergences have been removed at the arbitrary order μ . The matrix element still contains a residual cut-off dependence that vanishes for $r_c \rightarrow 0$:

$$\langle \Psi_{\text{LO}} | V_{\text{F}}^{(\mu)} + V_{\text{C}}^{(\mu)} | \Psi_{\text{LO}} \rangle = V_0 + V_a r_c^a + \dots, \quad (124)$$

where $V_{\text{C}}^{(\mu)}$ refers to the contact-range potential that renormalizes the order μ calculation, while V_0 and V_a are coefficients. The point is that the residual cut-off dependence indicates that the next new higher-order coupling of the contact-range potential enters at order $\mu + a$. What is the reason for that? Let us assume that the next divergence indeed enters at order $\mu + a$. The softest divergence that we are expected to find in the matrix elements of the potential is logarithmic, thus

$$\langle \Psi_{\text{LO}} | V_{\text{F}}^{(\mu+a)} + V_{\text{C}}^{\prime(\mu)} | \Psi_{\text{LO}} \rangle \propto \log r_c, \quad (125)$$

where the previous matrix element is the one for the finite-range potential at order $\mu + a$ plus the number of contact-range couplings that is expected at order μ . Notice that the value of these couplings change order-by-order, but their number only changes at the order at which a new C_{2n} coupling is included: we have written $V_{\text{C}}^{\prime(\mu)}$ with a prime to indicate this fact. As the divergence of the finite-range potential is $V_{\text{F}}^{(\nu)}(r) \sim 1/r^{3+\nu}$, it is not difficult to infer that going one order down translates into a residual cut-off dependence of r_c while going one order up gives rise to a $1/r_c$ divergence. Equivalently, if we move a orders down the expansion the residual cut-off dependence will be r_c^a , from which the previous conclusion about power counting follows.

After this an example might be the best way to illustrate the method. The easiest one is that of a contact-range theory with a large scattering length. If we

solve $k \cot \delta$ at LO for the delta-shell short-range potential that we have been using, we obtain the result

$$k \cot \delta = -\frac{1}{a_0} + \left[\frac{2}{3} r_c - \frac{1}{3} \frac{r_c^2}{a_0} \right] k^2 + \mathcal{O}(k^4), \quad (126)$$

where the residual cut-off dependence is of order r_c . The conclusion is that the next counterterm is one order below C_0 . That is, C_2 enters at NLO. If we now proceed to compute $k \cot \delta$ at NLO we findⁱ

$$k \cot \delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \left[\frac{1}{6} r_0 r_c^2 + \mathcal{O}(r_c^3) \right] k^4 + \mathcal{O}(k^6). \quad (127)$$

The residual cut-off dependence is now of order r_c^2 : C_4 enters two orders below C_2 , that is N³LO. Strictly speaking residual cut-off dependence is a constructive process and we can use it to determine the location of only the next coupling that enters in the theory, but not more. If we want to find the order of C_6 we must first compute the N³LO amplitudes that contain C_4 and from this extract the residual cut-off dependence. Alternatively, we can always rely on the natural expectation that C_6 should enter two orders below C_4 .

Finally it is interesting to check the predictions of this idea for the tensor force. On general grounds we expect the cut-off dependence of a LO calculation of the phase shift with attractive tensor OPE to be¹⁵

$$\frac{d}{dr_c} \delta_{\text{LO}} \propto r_c^{3/2}, \quad (128)$$

which after integration leads to a residual dependence of $r_c^{5/2}$. This indicates that C_2 enters at N^{5/2}LO in agreement with the previous determinations.

2.6. Power Counting and Wilsonian Renormalization

The central point of this section has been to review how we can derive power counting in Wilsonian renormalization. The application of renormalization group analysis (RGA) to nuclear EFT, though sometimes considered a bit arcane, can lead to interesting insights. To illustrate the idea we have taken non-relativistic s-wave scattering as an example and shown in detail how to derive well-known facts about power counting in the two-body sector that we review in Table 2.2. We have used two equivalent RG formulations. The first is the standard one in which the starting point is a “fundamental theory”: we include a cut-off in the theory and then evolve it from the ultraviolet to the infrared. As a result we find a physical theory — the EFT — that is equivalent to the fundamental theory at low energies. The EFT incorporates the familiar counting rules that we already know, for instance the enhancement of the couplings when the scattering length is large. The second is a

ⁱThe NLO calculation includes C_2 at first order perturbation theory. Otherwise the residual cut-off dependence will be different.

more streamlined formulation in which we do not directly evolve the EFT from the fundamental theory and instead use a convenient shortcut to determine the size of the effective couplings. This shortcut is the calculation of the anomalous dimension of the couplings, which turns out to be relatively easy, at least in the two-body case. As we will see, this is also the case for reactions of external probes on two-body states and for the three-body problem in pionless.

Other important point is the relationship between RGA and more standard techniques of determining the power counting. By more standard techniques we refer to ultraviolet renormalization and residual cut-off dependence. In principle we expect all derivations to be equivalent. In practice this equivalence has to be shown by means of concrete calculations. The results indicate the direct equivalence with RGE in the absence of singular pion exchanges, i.e. in the absence of the tensor force. If the tensor force is present there is an apparent contradiction though: the C_{2n} couplings seem to be more demoted in ultraviolet renormalization than in RGA or in residual cut-off dependence. This disagreement can be explained as a contamination of the C_{2n} couplings with the Λ_{NN} scale. In other words, what we call C_{2n} in ultraviolet renormalization is not really the C_{2n} coupling, but rather a coupling with the structure $E_{2n} k^{2n} \Lambda_{\text{NN}}^{n/2}$ instead of the expected $C_{2n} k^{2n}$. This distinction is in fact analogous to the one that is usually made between C_0 and $D_2 m_\pi^2$.

There are a few open problems that we have not addressed though. The most obvious example is the power counting of the triplet channels where the OPE potential is a repulsive $1/r^3$. The RG evolution of the couplings indicates that, with the exception of C_0 , the scaling of the couplings is identical to that of the attractive $1/r^3$ potential. This conclusion agrees with a previous RGA of the OPE potential,¹⁴ but it is counterintuitive to say the least. If the long-range physics is repulsive we expect that the short-range physics will play a lesser role at low energies because the repulsive long-range physics acts as a potential barrier. That is why some authors prefer to use naive dimensional analysis in this case.^{26,28} Other problem that is related to the previous one is what happens with coupled channels such as the $^3S_1 - ^3D_1$ deuteron channel. In this latter case three different countings have been proposed.^{14,25,28}

Even in the attractive triplet channels there are two proposals about the scaling of the C_0 coupling: does it enter at LO (Q^{-1}) or at $N^{1/2}$ LO ($Q^{-1/2}$)? The RGA of Birse¹⁴ assumes that C_0 is $N^{1/2}$ LO in the attractive triplet channels. But here we have taken the view that C_0 is LO: it has to be there because the LO wave function of a non-perturbative attractive triplet is not well defined without the inclusion of short-range physics. We find it worth noticing that the perturbation δC_0 of this coupling is $N^{1/2}$ LO($Q^{-1/2}$), which is where C_0 is predicted to be by Birse's RGA.¹⁴ That is, the cause of the disagreement seems to be that Ref. 14 overlooks the presence of short-range physics in the LO wave functions: the C_0 coupling is implicit in the choice of a semiclassical phase, i.e. the choice of ϕ_3 in Eq. (57).

One last problem is the scaling of the C_{2n} couplings in the 1S_0 singlet. The

standard counting^{37,46} is that the piece of C_{2n} that carries physical information enters at $N^{2n-1}\text{LO}$: C_2 at NLO, C_4 at $N^3\text{LO}$, C_6 at $N^5\text{LO}$, etc. Long and Yang²⁹ get to a different conclusion instead: the C_{2n} 's enter at $N^n\text{LO}$: C_2 at NLO, C_4 at $N^2\text{LO}$, C_6 at $N^3\text{LO}$, etc. The conclusion is a bit puzzling: according to Ref. 29, dimensional regularization with minimal subtraction leads to a stronger enhancement of the C_{2n} couplings than cut-off regularization

Other aspect to discuss is the interpretation of the cut-off in EFT. The RG equations use a cut-off in the region $M \geq \Lambda \geq Q$. This raises the question of whether the cut-off should stay below the breakdown scale, as happens in the RGA. The answer is *not necessarily*. The RG equations are formulated with the limits $M \geq \Lambda$ and $\Lambda \geq Q$ in mind to uncover the scaling of the couplings. This is a formal requirement to make the analysis easier, not a practical requirement in EFT calculations. EFTs are RG-invariant: the cut-off does not appear in the observable quantities that we compute, only in the intermediate calculations leading to the EFT predictions. That is, the cut-off is kept low in RGA with the intention of making the scaling of the contact-range couplings as obvious as possible. Once the RGA is done the only constraints about the size of the cut-off are practical ones.

One of these constraints is the existence of residual cut-off independence. In most EFT calculations we do not include all the couplings that are required to achieve exact RG independence. We only include the couplings that carry physical information at the order we are considering. There are two reasons for doing this: first, exact RG independence is not well-defined if we are making calculations in an EFT at a given order. The systematic EFT error is always present and RG independence must be understood within this error. The second reason is that exact RG independence requires the inclusion of what we have called here the redundant couplings, i.e. the C_{2n}^R piece of the couplings in the RG equations. These redundant operators can be calculated and included explicitly in a few specific cases: the KSW counting^{37,38} and pionless EFT with PDS.⁴⁶ But on more general cases this is unpractical and not really necessary. It is easier to keep the residual cut-off dependence under control by a judicious choice of the cut-off. For this condition to be true it is usually enough for Λ to be of the order of the hard scale, though the exact details will depend on the regulator.

Other important thing is to stress that a power counting is merely an ideal organization of the size of the interactions of a theory. They are derived under the assumption that the scale separation is large and that we can clearly classify all scales either as soft (Q) or hard (M). However the real physical world is not necessarily like that. What do we do if we have a two-body system with a scattering length that is neither small nor large? The point is that what we obtain with RGA is just an approximation to a more complex reality. In particular other power countings are possible beyond the ones we have discussed here. For instance in Ref. 43 van Kolck developed a counting for two-body systems in which the scattering length is tiny. Other possibility is when both the scattering length and the effective range are

large, a case which can be useful for the description of low lying s-wave resonances or even for the 1S_0 singlet to improve the convergence.⁴⁷ That means that we are entitled to curb the counting rules in view of practical physical information of the system. The limit is theoretical consistency: the EFT must be equivalent to the fundamental theory at low energies, which means that renormalizability must be respected.

3. Beyond the Two-Body Problem

The principles of renormalization work in the same way for operators different than the two-body potential. The advantage of calculating the anomalous dimension is that we can extend the idea seamlessly to any other problem. The point is to have a coupling and a polynomial contact-range operator, to evaluate their matrix element and to demand RG invariance at the end,

$$\frac{d}{d\Lambda} \langle \Psi'_{\text{LO}} | O_C | \Psi_{\text{LO}} \rangle = 0. \quad (129)$$

Actually the whole process amounts to nothing more than following a recipe. The only thing we have to do is to choose the wave functions and the operators that are appropriate for the particular physical process we are studying.

3.1. External Probes and Power Counting

Let us consider the case of a reaction involving a external probe and the deuteron (or more generally the two-nucleon system). In this case the contact-range operators of the theory involve two nucleons and one (or more) external fields. The external fields we are interested in are pions, photons and neutrinos. In principle the initial and final wave functions are the product of a two-nucleon wave function and the wave function of zero, one or more external probes

$$|\Psi_{\text{EFT}}\rangle = |\Psi_{\text{NN}}, \{\phi_i(\vec{q}_i)\}\rangle, \quad (130)$$

where $\{\phi_i\}$ refers to the probes, with the index $i = 0, 1 \dots n$. The contact-range operator involves a coupling and a polynomial of the momenta of the nucleons and the external probes. In the plane wave basis it reads

$$\langle \vec{p}', \{\phi'_j(\vec{q}_j')\} | O_C | \vec{p}, \{\phi_i(\vec{q}_i)\} \rangle = C(\Lambda) \times P_\Lambda(\vec{p}, \vec{p}', \vec{q}_i, \vec{q}_j'), \quad (131)$$

where \vec{p} (\vec{p}') is the center-of-mass momentum of the initial (final) two-nucleon system and \vec{q}_i (\vec{q}_j') the momenta of all the incoming (outgoing) probes involved in the operator. In general the polynomial P_Λ will involve spin and isospin degrees of freedom, but for the moment we will ignore them.

How does one evaluate the matrix elements? If we consider that the wave functions of the external probes are plane waves, the evaluation of the contact-range operator yields

$$\langle \Psi'_{\text{EFT}} | O_C | \Psi_{\text{EFT}} \rangle = C(\Lambda) \langle \Psi'_{\text{NN}} | P_\Lambda(\vec{p}, \vec{p}', \vec{q}_i, \vec{q}_j') | \Psi_{\text{NN}} \rangle, \quad (132)$$

which at the end involves the matrix element of a polynomial between the initial and final two-nucleon wave functions. That is, we end up with the same type of matrix elements as in two-nucleon scattering. The calculation of the RGE is done as in the case of two-nucleon scattering, except for two differences. The first difference is that the polynomial contains new pieces that were not present before: the momenta of the external probes. However they factor out of the matrix element and do not contribute to the RGE evolution of the couplings. The second difference is that the initial and final two-nucleon wave functions can be different. While in two-nucleon scattering the initial and final states involve the same scattering channel (1S_0 , $^3S_1 - ^3D_1$, 3P_0 ...), this is not true in general for a reaction. The reason is that the external probe carries quantum numbers, which means that the scattering channel can change in the reaction. We can have transitions from $^3S_1 - ^3D_1$ to 1S_0 and other combinations. As a consequence the RGE ends up being

$$\frac{d}{dr_c} \left[\frac{u(r_c)u'(r_c)}{r_c^2} C(r_c) + \dots \right] = 0, \quad (133)$$

where u and u' are the reduced wave functions of the initial and final states.

The evaluation of the anomalous dimension for a coupling C depends on the channels involved in the reaction. If we consider a short-range theory (or a theory containing a long-range potential that is perturbative), the 1S_0 and 3S_1 partial waves behave in exactly the same way. The outcome is an enhancement of M^2/Q^2 in any contact-range operator involving these partial waves. If we consider a pionful EFT with non-perturbative pions the enhancement depends on the partial waves involved: the presence of the 1S_0 partial wave will elicit a M/Q enhancement, while the 3S_1 a $(M/Q)^{1/4}$. These factors must be multiplied: a $^3S_1 \rightarrow ^3S_1$ transition involves a $(M/Q)^{1/4} \times (M/Q)^{1/4} = (M/Q)^{1/2}$ enhancement, a $^3S_1 \rightarrow ^1S_0$ a $(M/Q)^1 \times (M/Q)^{1/4} = (M/Q)^{5/4}$ and a $^1S_0 \rightarrow ^1S_0$ a $(M/Q)^1 \times (M/Q)^1 = (M/Q)^2$. In short, the result is surprisingly simple.

The extension to P-waves, though not derived in the present review, is worth a brief comment: while in pionless they do not entail any enhancement, in pionful a transition involving a 3P_0 partial wave will generate a $(M/Q)^{5/4}$ enhancement^j. In contrast the 1P_1 does not involve any enhancement, the 3P_1 most probably not (though it is an instance of a repulsive $1/r^3$ and thus open to discussion) and the 3P_2 probably generates the same enhancement as the 3P_0 . Putting the pieces together we see that the two transitions that will be most enhanced in pionful are $^1S_0 \rightarrow ^3P_0$ and $^1S_0 \rightarrow ^3P_2$ by a factor $(M/Q)^1 \times (M/Q)^{5/4} = (M/Q)^{9/4}$, followed by $^3S_1 \rightarrow ^3P_0$ and $^3S_1 \rightarrow ^3P_2$ by a factor $(M/Q)^{1/4} \times (M/Q)^{5/4} = (M/Q)^{3/2}$. In electromagnetic processes $^1S_0 \rightarrow ^3P_0$ is forbidden, but the others can appear as magnetic quadrupole and electric dipole transitions. Curiously these S- to P-wave

^jFor P-waves the enhancements are bigger than in S-waves. The reason is that the P-wave contacts are initially more suppressed, which also means that there is more room for increasing the size of the couplings.

transitions can be very interesting if one considers parity violation^{48,49} or parity plus time-reversal violation, in which case they might contribute to the electric dipole moments of light nuclei⁵⁰ (either with external probes or as a part of the potential).

3.2. Electroweak Reactions on the Deuteron

The application of the previous ideas to electroweak reactions is mostly direct except for the symmetry constraints of each particular case. Electromagnetic processes respect gauge symmetry and as a consequence also charge conservation. This will have an impact on which are the allowed contact two-body currents in a reaction. The lagrangian interaction term of a reaction involving a single photon takes the general form $A_\mu J^\mu$, with A_μ the photon field and J^μ the electromagnetic current and $\mu = 0, 1, 2, 3$ a Lorentz index. For a matrix element involving initial and final two-nucleon states that are on the mass shell, J^μ obeys the Ward identity

$$q_\mu \langle \Psi' | J^\mu(q) | \Psi \rangle = 0. \quad (134)$$

We can distinguish between two pieces of the current: the longitudinal piece, which is parallel to the moment of the photon \vec{q} , and the transversal piece, which is perpendicular to \vec{q} . Reactions where the external probe is a photon — deuteron photodisintegration and radiative capture of neutron by protons ($\gamma d \rightarrow np$ and $np \rightarrow d\gamma$) — depend on the transversal part of the current and are not constrained by gauge symmetry. However there is the indirect constraint that the two-body current operator is transversal to the photon momentum, which entails that the lowest dimensional operator we can build is

$$\langle \vec{p}' | \vec{J}_{2B}^T(\vec{q}) | \vec{p} \rangle = M(\Lambda) \vec{\beta} \times \vec{q}, \quad (135)$$

where β a pseudovector that encodes the spin and isospin dependence. The operator contains one power of the external momentum \vec{q} , that is, one power of Q . For determining at which order in the EFT expansion this operator enters we have to compare with the one-body current operator, which in the Breit frame^k reads

$$\langle \vec{p}' | \vec{J}_{1B}(\vec{q}) | \vec{p} \rangle = \left[e \frac{\vec{p}' + \vec{p}}{2M_N} + i \hat{\mu}_B \times \vec{q} \right] \delta(\vec{p}' - \vec{p} - \frac{\vec{q}}{2}), \quad (136)$$

that scales as Q^{-2} (because the Dirac delta counts as Q^{-3}). In the expression above e is the charge of the two-nucleon state, \vec{p} and \vec{p}' the two-nucleon center-of-mass initial and final momenta, M_N the nucleon mass and $\hat{\mu}_B$ the magnetic moment operator. In NDA the coupling $M(\Lambda)$ scales as $1/M^4$ and the two-body contact-range current enters at N³LO compared to the one-body current. However in nuclear

^kThe Breit frame is equivalent to taking the zero component of the photon quadrimomentum equal to zero, i.e. there is no energy transfer. In more practical terms this means that if we have an incoming photon with 3-momentum \vec{q} then the total 3-momentum of the incoming and outgoing two-nucleon system is $\vec{P} = -\vec{q}/2$ and $\vec{P}' = +\vec{q}/2$.

physics this two-body current is sandwiched between a 1S_0 and 3S_1 partial wave and that changes the anomalous dimension of the coupling. As a consequence of the enhancements that we analyzed previously, in pionless⁴⁶ the contact current enters at NLO, while in pionful¹⁶ enters at $N^{7/4}$ LO.

Now we consider the deuteron form factors. They are important for elastic electron-deuteron scattering, which is a reaction mediated by a virtual photon. That is, the longitudinal as well as the transversal current will have to be taken into account. The deuteron is not a point particle: its response to a virtual photon is described with form factors. As the angular momentum of the deuteron is $J = 1$ there are three independent form factors¹: the charge, the magnetic and the quadrupole form factor. In the Breit frame they are defined as^{51–53}

$$G_C(\vec{q}) = \frac{1}{3e} \sum_{m_d=-1}^{+1} \langle \Psi_d(1m_d) | J_0(\vec{q}) | \Psi_d(1m_d) \rangle, \quad (137)$$

$$G_M(\vec{q}) = -\frac{1}{e\sqrt{2}\eta} \langle \Psi_d(11) | J_1(\vec{q}) + iJ_2(\vec{q}) | \Psi_d(10) \rangle, \quad (138)$$

$$G_Q(\vec{q}) = \frac{1}{2e\eta M_d^2} [\langle \Psi_d(10) | J_0(\vec{q}) | \Psi_d(10) \rangle - \langle \Psi_d(11) | J_0(\vec{q}) | \Psi_d(11) \rangle], \quad (139)$$

where $|\Psi_d(1m_d)\rangle$ refers to the deuteron wave function with the third component of the total spin being m_d , $\eta = Q^2/(4M_d^2)$ where $Q^2 = |\vec{q}|^2 - |q_0|^2$, with q the 4-momentum of the virtual photon and M_d is the deuteron mass. The 3-momentum of the photon is taken to be in the $i = 3$ direction, i.e. $\vec{q} = (0, 0, q)$. That is why the matrix elements of the J_3 component of the current are not considered above: these matrix elements are related to the matrix elements of J_0 by means of the Ward identity.

The charge and quadrupole form factors depend on the charge current J_0 . The lowest dimensional contact operator contributing to J_0 is in principle

$$\langle \vec{p}' | J_{2B}^0(\vec{q}) | \vec{p} \rangle = C(\Lambda), \quad (140)$$

but this operator is forbidden by charge conservation, the most direct consequence of gauge symmetry. The reason is that it gives a non-zero contribution to the deuteron charge. The lowest dimensional operators compatible with charge conservation are

$$\langle \vec{p}' | J_{2B}^0(\vec{q}) | \vec{p} \rangle = D(\Lambda) \vec{q}^2 + Q(\Lambda) \left[3(\vec{S} \cdot \vec{q})^2 - \vec{q}^2 \right], \quad (141)$$

with \vec{S} the spin operator of the deuteron. The couplings $D(\Lambda)$ and $Q(\Lambda)$ represent a direct contribution to the deuteron charge radius and quadrupole moment respectively. Their size is $1/M^4$ in NDA. The deuteron wave functions induce an anomalous dimension for $D(\Lambda)$ and $Q(\Lambda)$, which scale as $1/M^2 Q^2$ in pionless⁴⁶ and $1/M^{7/2} Q^{1/2}$ in pionful.¹⁶ The LO is set by the one-body charge current

$$\langle \vec{p}' | J^0(\vec{q}) | \vec{p} \rangle = e \delta(\vec{p}' - \vec{p} - \frac{\vec{q}}{2}), \quad (142)$$

¹Assuming parity and time-reversal invariance. Otherwise there will be more.

that scales as Q^{-3} . Therefore $D(\Lambda)$ and $Q(\Lambda)$ enter at $N^5\text{LO}$ in NDA,⁵⁴ $N^3\text{LO}$ in pionless⁴⁶ and $N^{9/2}\text{LO}$ in pionful.¹⁶

The magnetic form factor describes the response of the deuteron to a transversal current. It is analogous to the matrix elements that appear in the deuteron breakup reactions. The power counting of the contact currents is identical to that case except for the difference that the initial and final states are in the $^3S_1 - ^3D_1$ channel. In NDA⁵⁴ the contact contribution to G_M enters at $N^3\text{LO}$, in pionless⁴⁶ at NLO and in pionful¹⁶ at $N^{5/2}\text{LO}$.

To close the discussion we will briefly consider proton-proton fusion ($pp \rightarrow de^+\nu_e$), which is a weak process. The average momenta of protons at the core of the sun is about 1 keV. From the perspective of nuclear physics these are extremely low momenta and we expect solar proton-proton fusion to happen almost entirely via the s-wave transition $^1S_0 \rightarrow ^3S_1$. For this reaction the relevant one-body weak current is axial and takes the form:

$$\begin{aligned} \langle \vec{p}' | \vec{A}_{1B}(\vec{q}_{e^+} + \vec{q}_{\nu_e}) | \vec{p} \rangle &= -G_V g_A [\vec{\sigma}_1 \tau_1^- + \vec{\sigma}_2 \tau_2^-] \\ &\times \delta^{(3)}(\vec{p}' - \vec{p} - \frac{1}{2}\vec{q}_{e^+} - \frac{1}{2}\vec{q}_{\nu_e}), \end{aligned} \quad (143)$$

where G_V is the weak vector coupling, g_A the axial-to-vector ratio, \vec{q}_{e^+} and \vec{q}_{ν_e} are the momenta of the final positron and neutrino, σ_i the spin of the nucleon $i = 1, 2$ and τ_i^- an isospin operator that turns a proton into a neutron, i.e. $\tau^-|p\rangle = |n\rangle$. This one-body current scales as Q^{-3} owing to the delta. Meanwhile the simplest axial two-body current that we can construct takes the form

$$\langle \vec{p}' | \vec{A}_{1B} | \vec{p} \rangle = A(\Lambda) \vec{\beta}, \quad (144)$$

with β a pseudovector containing the spin and isospin components (the exact form of this current can be checked in Ref. 55). In NDA this two-body current scales as Q^0 , i.e. $N^3\text{LO}$ relative to the one-body current. In pionless and pionful the $A(\Lambda)$ coupling will be enhanced by a factor $(M/Q)^2$ and $(M/Q)^{5/4}$ respectively, that is, NLO and $N^{7/4}\text{LO}$.

3.3. The Three Body Contact in Pionless

Other application is the power counting of the three-body contact-range interaction in pionless EFT.^{56–59} A three-body system of identical bosons that interacts via two-body contact-range interactions can bind, more so if the two-body system contains a bound or virtual state. The wave function can be expressed as a sum of three components (the Fadeev components)

$$|\Psi_{3B}\rangle = \langle \vec{p}_1 \vec{k}_{23} | \psi_{3B} \rangle + \langle \vec{p}_2 \vec{k}_{31} | \psi_{3B} \rangle + \langle \vec{p}_3 \vec{k}_{12} | \psi_{3B} \rangle, \quad (145)$$

where all the components are identical: we have bosons and the wave function is invariant under permutation of the particle labels. The momenta \vec{p}_i , \vec{k}_{ij} are the

Jacobi momenta, which are

$$\vec{p}_1 = \frac{2}{3} \vec{q}_1 - \frac{1}{2} (\vec{q}_2 + \vec{q}_3), \quad (146)$$

$$\vec{k}_{23} = \frac{1}{2} (\vec{q}_2 - \vec{q}_3), \quad (147)$$

plus permutations, with $\vec{q}_{1,2,3}$ the momenta of each of the particles. For the case of s-wave, non-perturbative contact-range forces, there is a compact ansatz for the Fadeev component ψ_{3B}

$$\langle \vec{p} \vec{k} | \psi_{3B} \rangle = \frac{a(p)}{\frac{3}{4} p^2 + k^2 + \gamma_3^2} \quad (148)$$

where γ is the wave number of the three-body system and $a(p)$ is a function of the Jacobi momentum p . If the mass of each of the identical particles is M , the binding energy is $E_B = -\gamma_3^2/M$. The function $a(p)$ is given by

$$a(p) \propto \frac{1}{p^2} f(p), \quad (149)$$

at large momenta, where $f(p)$ is an oscillatory function of the type

$$f(p) = \sin \left[s_0 \log \frac{p}{p_0} \right], \quad (150)$$

where $s_0 \simeq 1.0064$ and p_0 is a reference momentum that we will discuss in a moment.

The details of how one reaches the three-body wave-function are irrelevant for our purposes here, but can be consulted in the literature.^{56,57} The relevant point here is that we can make interesting conclusions about the power counting from inspecting the three-body bound-state wave function. The value of the reference momentum p_0 cannot be determined from the two-body contact interactions unambiguously. If we include a cut-off Λ then we can calculate a $p_0(\Lambda)$, but it does not converge to a specific value as we increase Λ . That is, the value of p_0 depends on the short-range physics. But since the two-body short-range physics are already included in the EFT, the conclusion is that there is a contact-range three-body force also entering at LO.

Now we rederive this result with the calculation of the anomalous dimension of the three-body contact-range coupling. In momentum space the lowest order three-body contact-range potential reads

$$\langle \vec{p}', \vec{k}' | V_C | \vec{p}, \vec{k} \rangle = C_3, \quad (151)$$

where the naive estimation of its size is $C_3 \sim 1/M^3$. The matrix element of this potential when sandwiched between the Ψ_{3B} wave function is

$$\langle \Psi_{3B} | C_3 | \Psi_{3B} \rangle \propto \langle \psi_{3B} | C_3 | \psi_{3B} \rangle = C_3 \left[\int_{\Lambda} \frac{d^3 \vec{p}}{(2\pi)^3} \frac{d^3 \vec{k}}{(2\pi)^3} \frac{a(p)}{\frac{3}{4} p^2 + k^2 + \gamma^2} \right]^2, \quad (152)$$

which diverges as

$$\langle \Psi_{3B} | C_3 | \Psi_{3B} \rangle \propto C_3 \Lambda^4. \quad (153)$$

We end up with the RGE

$$\frac{d}{d\Lambda} [C_3(\Lambda) \Lambda^4 + \dots] = 0, \quad (154)$$

which implies a $(M/Q)^4$ enhancement over the NDA estimation. The non-relativistic three-body propagator counts as Q^4 :

$$\begin{aligned} I_3(E) &= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{d^3\vec{k}}{(2\pi)^3} G_0(E) \\ &= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{ME - M(\frac{3}{4}p^2 + k^2)} \sim Q^4, \end{aligned} \quad (155)$$

which at the end means that C_3 has to be iterated to all orders because $C_3 \sim C_3 I_3(E) C_3$. That is, C_3 enters at LO.

We can easily apply these ideas to the three-nucleon system, though there are a few complications owing to spin, isospin and the fact that nucleons are fermions. The presence of spin and isospin degrees of freedom allows the spatial part of the wave function to be partially or fully symmetric for specific configurations. When that happens the conclusions that we derived for three-boson system may apply to the three-nucleon system as well.⁵⁸ This is the case for the triton, where the three-nucleon contact enters at LO in the pionless EFT. This also happens for neutron-deuteron scattering in the spin-1/2 configurations (the doublet), for the simple reason that this is the same channel as the triton. Trivially this conclusion applies too for the ^3He nucleus and doublet proton-deuteron scattering at LO, though here we have the additional complication of Coulomb.⁶⁰ Recently it has been discovered that Coulomb is able to modify the counting at NLO,^{61,62} but only if it is treated non-perturbatively.⁶³ On the contrary for the spin-3/2 configurations (the quartet) of the three-nucleon system the size of the three-nucleon contact is the one expected in NDA. The reason is that the spatial wave function cannot be symmetric for the quartet. If we visualize the quartet as a nucleon scattering off a deuteron it is clear that this nucleon must be in a P-wave with respect to the deuteron. The wave function has a certain resemblance with the one we have studied for the three-boson system. However the piece that depends on \vec{p} is not of the type $a(p)$ but rather $\vec{\beta} \cdot \vec{p} a(p)$. In addition the power-law dependence of $a(p)$ is much more suppressed at high momenta, a change that modifies the anomalous dimension of the three-body coupling. The outcome is that the contact coupling is not enhanced (and might even be demoted^m) in this case.⁵⁹

The previous ideas can be extended to triton and ^3He reactions. As we already explained, in a reaction the few-nucleon part of the wave function factors out when computing the matrix elements. The conclusion is that contact few-body operators

^mHere we will not consider the possibility of the demotion of a coupling. The reason is that even if the anomalous dimension of a coupling is positive, this only induces small corrections over the $1/M^d$ baseline value of the coupling that is used as a boundary condition at $\Lambda = M$.

involving external probes have the same type of enhancements as contact few-body forces. On practical terms a three-body operator involving the triton (or the doublet) in the initial and final states will be enhanced by the factor $(M/Q)^4$. If one of the initial and final channels is the triton/doublet and the other is the quartet, the enhancement will be $(M/Q)^2$. Finally a reaction only involving an initial and final quartet is not expected to be enhanced.

4. Conclusions

In this manuscript I have attempted to illustrate the application of Wilsonian renormalization to nuclear EFT. The starting point is the non-relativistic two-body scattering problem. The requirement that the scattering amplitude is invariant under changes in the cut-off generates RGEs for the couplings of the EFT. The RGEs can be calculated easily from the two-body wave functions. The solution of the RGEs in the infrared — as we change the momentum cut-off from M to Q — determines the power counting of the EFT. There are different power countings depending on the initial assumptions about scattering at low energies. In agreement with the previous literature, for non-relativistic two-body systems that interact via a regular potential there are two general power countings, the natural and the unnatural one.^{12,13} If the potential is non-perturbative and singular, as happens in nucleon-nucleon scattering in the triplet partial waves, there is a unique power counting and the idea of fine tuning is not that crucial: all the scattering lengths are equally fine-tuned. Yet this conclusion is not universally agreed upon within the context of RGA.¹⁴

There is also the interesting observation that the solution of the RGEs for a particular coupling is connected to its anomalous dimension.¹⁶ The anomalous dimension refers to how a coupling changes under a rescaling of the cut-off. It turns out that the calculation of the anomalous dimension is trivial, merely involving the evaluation of a matrix element between the EFT wave functions. Incidentally this development also makes it easy to determine power counting beyond the two-body system. We have also shown the equivalence among Wilsonian renormalization, standard or ultraviolet renormalization and the analysis of the residual cut-off dependence as methods to uncover the power counting. But there are also open problems regarding the renormalization of singular interactions. For attractive singular potentials the previous equivalence is not completely proven yet (which explains the existence of slightly different versions of the power counting in the literature^{14,25–28}), though the evidence pointing towards this direction is convincing. In the case of a repulsive singular potential the big unsolved issue is whether its power counting is the same as for the attractive case or if it follows NDA. This question has so far evaded a satisfactory analysis.

The advantage of the anomalous dimension is that it can be easily calculated in the case of reactions of external electroweak probes acting on the two-nucleon system. The power counting is essentially the one of the two-nucleon system for the simple reason that the probes we are considering can be described with a plane

wave and factored out of the RGEs. As a consequence this idea does not hold if the external probe is a third nucleon. For the three-nucleon system we must calculate the wave functions again if we want to be able to determine the power counting. The behaviour of the three-body wave functions is well-known for a contact-range potential, from which we can independently reproduce the power counting of pionless EFT for the triton and neutron-deuteron scattering. In the future once the power-law behaviour of the wave functions of the triton is properly analyzed it will be possible to determine the power counting of the three-nucleon system in pionful EFT (though there are preliminary results⁶⁴).

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Appendix A. The Delta-Shell Potential

In this appendix we derive the solution of the Schrödinger equation for a delta-shell potential. The starting point is

$$-u_k'' + 2\mu [V(r; r_c) + V_C(r; r_c)] u_k = k^2 u_k, \quad (\text{A.1})$$

where u_k is the reduced wave function, k the momentum, V the finite-range potential and V_C the delta-shell potential. The form of V_C is

$$V_C(r; r_c) = \frac{C_k(r_c)}{4\pi r_c^2} \delta(r - r_c), \quad (\text{A.2})$$

where $C_k(r_c)$ can be expanded in powers of k^2 . We can see that V_C only acts at $r = r_c$. Now we integrate the Schrödinger equation in the vicinity of r_c

$$\int_{r_c-\epsilon}^{r_c+\epsilon} (-u_k'' + 2\mu [V(r; r_c) + V_C(r; r_c)] u_k) dr = k^2 \int_{r_c-\epsilon}^{r_c+\epsilon} u_k(r) dr, \quad (\text{A.3})$$

with ϵ a small positive number. The evaluation of the following pieces is direct

$$\int_{r_c-\epsilon}^{r_c+\epsilon} u_k''(r) dr = u_k'(r_c + \epsilon) - u_k'(r_c - \epsilon), \quad (\text{A.4})$$

$$2\mu \int_{r_c-\epsilon}^{r_c+\epsilon} V_C(r; r_c) u_k(r) dr = 2\mu \frac{C_k(r_c)}{4\pi r_c^2} u_k(r_c), \quad (\text{A.5})$$

while the two remaining pieces vanish in the $\epsilon \rightarrow 0$ limit

$$\lim_{\epsilon \rightarrow 0} \int_{r_c-\epsilon}^{r_c+\epsilon} 2\mu V(r; r_c) u_k(r) dr = 0, \quad (\text{A.6})$$

$$\lim_{\epsilon \rightarrow 0} k^2 \int_{r_c-\epsilon}^{r_c+\epsilon} u_k(r) dr \rightarrow 0, \quad (\text{A.7})$$

the reason being that the integrand is bounded in the region around r_c . We can see that while u_k is continuous at $r = r_c$, u'_k develops a discontinuity. Putting the pieces together for $\epsilon \rightarrow 0$ we arrive at

$$\frac{u'_k(r_c^+)}{u_k(r_c^+)} - \frac{u'_k(r_c^-)}{u_k(r_c^-)} = 2\mu \frac{C_k(r_c)}{4\pi r_c^2} u_k(r_c). \quad (\text{A.8})$$

Finally, expanding C_k in powers of k^2 we obtain Eq. (9).

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